New Identities for Degrees of Syzygies in Numerical Semigroups

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Abstract

We derive a set of polynomial and quasipolynomial identities for degrees of syzygies in the Hilbert series $H(\mathbf{d}^m;z)$ of nonsymmetric numerical semigroups $\mathsf{S}(\mathbf{d}^m)$ of arbitrary generating set of positive integers $\mathbf{d}^m=\{d_1,\ldots,d_m\}$, $m\geq 3$. These identities were obtained by studying together the rational representation of the Hilbert series $H(\mathbf{d}^m;z)$ and the quasipolynomial representation of the Sylvester waves in the restricted partition function $W(s,\mathbf{d}^m)$. In the cases of symmetric semigroups and complete intersections these identities become more compact.

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1 Introduction

The study of Diophantine equations is on the border-line between combinatorial number theory and commutative algebra. Most important results are awaiting at intersection of both theories that has been already seen in the last decades [25].

Focusing on such intersection in study of linear Diophantine equations, in this paper we bring together two different approaches, theory of restricted partition and theory of commutative semigroup rings, and show that such merging is fruitful to produce new results. It allows to establish a set of new quasipolynomial identities for degrees of the syzygies in the Hilbert series $H(\mathbf{d}^m;z)$ of nonsymmetric numerical semigroups $\mathsf{S}(\mathbf{d}^m)$ of any embedding dimension, $m\geq 3$, and arbitrary generating set of positive integers $\mathbf{d}^m=\{d_1,\ldots,d_m\}$ where $\gcd(d_1,\ldots,d_m)=1$. The special cases of symmetric semigroups and complete intersections make these identities more compact.

Regarding the novelty of these identities, to the best of our knowledge they have not been discussed earlier in literature. On the other hand, all necessary technical tools to derive them

were already elaborated in seminal works of Sylvester on partitions (1857, 1882, 1897) and in the basis theorem (1888) and the syzygy theorem (1890) of Hilbert.

The paper is organized in six sections. In section 2 we recall the main facts about numerical semigroups $\mathsf{S}(\mathbf{d}^m)$ and their Hilbert series $H(\mathbf{d}^m;z)$. For degrees of the syzygies we state the main result on polynomial and trigonometric identities (Theorem 1 and Corollary 1) which are independent of structure of the generating set $\mathbf{d}^m$. Another result on quasipolynomial identities (Theorem 2) is valid when among the generators d_i of the set $\mathbf{d}^m$ there exists a subset $\Xi_q(\mathbf{d}^m) \subset \mathbf{d}^m$ such that $\Xi_q(\mathbf{d}^m) := \{d_i \quad | \quad q \mid d_i\}$ and $\#\Xi_q(\mathbf{d}^m) \geq 2$.

In section 3 we recall the main facts about partition of nonnegative integer s into positive integers $\{d_1,\ldots,d_m\}$, each not greater than s, and about the number of nonequivalent partitions, or representations (Reps) $W(s,\mathbf{d}^m)$. The main emphasis is done on the Sylvester waves and their symbolic Reps which are useful to find new identities when applying the parity claims to the Sylvester waves.

In section 4 we derive the quasipolynomial Reps of Sylvester waves with trigonometric functions as coefficients and find when their leading terms are vanishing (Lemma 1).

In section 5 we derive the relationship between the rational Rep of the Hilbert series $H(\mathbf{d}^m;z)$ and the quasipolynomial Rep of the Sylvester waves in the restricted partition function $W(s,\mathbf{d}^m)$. This allows to prove Theorem 1 (in section 5.1) and Theorem 2 (in section 5.3) on polynomial and quasipolynomial identities for degrees of syzygies.

In section 6 we discuss different applications of Theorems 1 and 2 to the various kinds of numerical semigroups: complete intersections, symmetric semigroups (not complete intersections) generated by 4 and 5 elements, nonsymmetric and pseudosymmetric semigroups generated by 3 elements, and semigroups of maximal embedding dimension. We illustrate a validity of identities by examples for numerical semigroups which were discussed earlier in literature.

2 Numerical Semigroups and Hilbert Series $H(\mathbf{d}^m;z)$

Throughout the article we assume that the numerical semigroup $\mathsf{S}(\mathbf{d}^m)$ is generated by a minimal set of positive integers $\mathbf{d}^m = \{d_1,\ldots,d_m\}$ with finite complement in $\mathbb{N}$, $\#\{\mathbb{N}\setminus\mathsf{S}(\mathbf{d}^m)\} < \infty$. We study the generating function $H(\mathbf{d}^m;z)$ of such semigroup $\mathsf{S}(\mathbf{d}^m)$,

$$H(\mathbf{d}^m;z) = \sum_{s\,\in\,\mathsf{S}(\mathbf{d}^m)} z^s\,, \tag{2.1}$$

which is referred to as *the Hilbert series* of $\mathsf{S}(\mathbf{d}^m)$.

Recall the main definitions and facts on numerical semigroups which are necessary here. A semigroup $\mathsf{S}(\mathbf{d}^m) = \{s \in \mathbb{N}\cup\{0\} \mid s = \sum_{i=1}^m x_i d_i,\ x_i \in \mathbb{N}\cup\{0\}\}$, is said to be generated by *minimal set* of m natural numbers $d_1 < \ldots < d_m$, $\gcd(d_1,\ldots,d_m) = 1$, if neither of its elements is linearly

representable by the rest of elements. It is classically known that $d_1 \geq m$ [20] where d_1 and m are called *the multiplicity* and *the embedding dimension (edim)* of the semigroup, respectively. If equality $d_1 = m$ holds then the semigroup $\mathsf{S}(\mathbf{d}^m)$ is called of *maximal edim*. *The conductor* $c(\mathbf{d}^m)$ of semigroup $\mathsf{S}(\mathbf{d}^m)$ is defined by $c(\mathbf{d}^m) := \min\{s \in \mathsf{S}(\mathbf{d}^m) \mid s + \mathbb{N} \cup \{0\} \subset \mathsf{S}(\mathbf{d}^m)\}$ and related to *the Frobenius number* of semigroup, $F(\mathbf{d}^m) = c(\mathbf{d}^m) - 1$.

A semigroup $\mathsf{S}(\mathbf{d}^m)$ is called *symmetric* if for any integer s the following condition holds: if $s \in \mathsf{S}(\mathbf{d}^m)$ then $F(\mathbf{d}^m) - s \notin \mathsf{S}(\mathbf{d}^m)$. Otherwise $\mathsf{S}(\mathbf{d}^m)$ is called nonsymmetric. Notably that all semigroups $\mathsf{S}(d_1, d_2)$ are symmetric.

Denote by $\Delta(\mathbf{d}^m)$ the complement of $\mathsf{S}(\mathbf{d}^m)$ in $\mathbb{N}$, i.e., $\Delta(\mathbf{d}^m) = \mathbb{N} \setminus \mathsf{S}(\mathbf{d}^m)$, and call it the set of gaps. The cardinality ($\#$) of $\Delta(\mathbf{d}^m)$ is called *the genus* of $\mathsf{S}(\mathbf{d}^m)$, $G(\mathbf{d}^m) := \#\Delta(\mathbf{d}^m)$. For the set $\Delta(\mathbf{d}^m)$ introduce the generating function $\Phi(\mathbf{d}^m; z)$ which is related to the Hilbert series,

$$\Phi(\mathbf{d}^m; z) = \sum_{s \,\in\, \Delta(\mathbf{d}^m)} z^s \,, \qquad \Phi(\mathbf{d}^m; z) + H(\mathbf{d}^m; z) = \frac{1}{1-z} \,.$$

The Hilbert series $H(\mathbf{d}^m; z)$ of numerical semigroup $\mathsf{S}(\mathbf{d}^m)$ is a rational function [31]

$$H(\mathbf{d}^m; z) = \frac{Q(\mathbf{d}^m; z)}{\prod_{i=1}^{m}(1 - z^{d_i})} \,, \tag{2.2}$$

where $H(\mathbf{d}^m; z)$ has a pole $z = 1$ of order 1. The numerator $Q(\mathbf{d}^m; z)$ is a polynomial in z,

$$Q(\mathbf{d}^m; z) = 1 - Q_1(\mathbf{d}^m; z) + Q_2(\mathbf{d}^m; z) - \ldots + (-1)^{m-1} Q_{m-1}(\mathbf{d}^m; z) \,, \tag{2.3}$$

$$Q_i(\mathbf{d}^m; z) = \sum_{j=1}^{\beta_i(\mathbf{d}^m)} z^{C_{j,i}} \,, \quad 1 \leq i \leq m-1 \,, \quad \deg Q_i(\mathbf{d}^m; z) < \deg Q_{i+1}(\mathbf{d}^m; z) \,. \tag{2.4}$$

In formula (2.4) the positive numbers $C_{j,i}$ and $\beta_i(\mathbf{d}^m)$ denote *the degree of the syzygy* and *the Betti number*, respectively. The latter satisfy the equality [31]

$$\beta_0(\mathbf{d}^m) - \beta_1(\mathbf{d}^m) + \beta_2(\mathbf{d}^m) - \ldots + (-1)^{m-1}\beta_{m-1}(\mathbf{d}^m) = 0 \,, \quad \beta_0(\mathbf{d}^m) = 1 \,. \tag{2.5}$$

The summands $z^{C_{j,i}}$ in (2.4) stand for the syzygies of different kinds and $C_{j,i}$ are the degrees of homogeneous basic invariants for the syzygies of the ith kind,

$$\begin{aligned} &C_{j,i} \in \mathbb{N} \,, \quad C_{j+1,i} \geq C_{j,i} \,, \quad C_{\beta_{i+1},i+1} > C_{\beta_i,i} \,, \quad C_{1,i+1} > C_{1,i} \,, \quad \text{and} \\ &C_{j,i} \neq C_{r,i+2k-1} \,, \quad 1 \leq j \leq \beta_i(\mathbf{d}^m) \,, \quad 1 \leq r \leq \beta_{i+2k-1}(\mathbf{d}^m) \,, \quad 1 \leq k \leq \left\lfloor \frac{m-i}{2} \right\rfloor . \end{aligned} \tag{2.6}$$

The last requirement (2.6) means that all necessary cancellations (annihilations) of terms $z^{C_{j,i}}$ in (2.3) are already performed. However the other equalities, $C_{j,i} = C_{r,i+2k}$ and $C_{j,i} = C_{q,i}$, $j \neq q$, are not forbidden excluding the syzygy degrees of the last $(m-1)$th kind ([2], Lemma 2).

A degree of the polynomial $Q(\mathbf{d}^m; z)$ is strongly related [2] to the Frobenius number by

$$\deg Q(\mathbf{d}^m; z) = F(\mathbf{d}^m) + \sigma_1 \,, \quad \text{where} \quad \sigma_1 = d_1 + \ldots + d_m \,. \tag{2.7}$$

We present here the Hilbert series for nonsymmetric semigroup generated by triple $\{3,5,7\}$,

$$H\left(\{3,5,7\};z\right)=\frac{1-z^{10}-z^{12}-z^{14}+z^{17}+z^{19}}{\left(1-z^3\right)\left(1-z^5\right)\left(1-z^7\right)}\ . \tag{2.8}$$

For the rigorous notions of syzygies of the 1st and higher kinds, their moduli and specifying homomorphisms as well as the Betti numbers and minimal free resolution we refer to the book [12]. An informal description of syzygies, difference binomials and other homogeneous bases for the higher syzygies, which came by applying the Hilbert basis and syzygy theorems, can be found in the review [31]. Regarding the degrees of the syzygies, in the general case of nonsymmetric numerical semigroups $\mathsf{S}\left(\mathbf{d}^m\right)$, their values $C_{j,i}$ are usually obtained by computational algorithm [10] calculating the homogeneous bases for the syzygies moduli and their specifying homomorphisms in a minimal free resolution.

If a semigroup $\mathsf{S}\left(\mathbf{d}^m\right)$ is symmetric then a duality relation for numerator $Q\left(\mathbf{d}^m;z\right)$ holds [2]

$$Q\left(\mathbf{d}^m;\frac{1}{z}\right)z^{\deg Q(\mathbf{d}^m;z)}=(-1)^{m-1}Q\left(\mathbf{d}^m;z\right)\ , \tag{2.9}$$

and by consequence of (2.9) we have

$$\beta_k\left(\mathbf{d}^m\right)=\beta_{m-k-1}\left(\mathbf{d}^m\right)\ ,\quad C_{j,k}+C_{j,m-k-1}=\deg Q\left(\mathbf{d}^m;z\right)\ ,\quad \left\{\begin{array}{c} 0\le k<m, \\ 1\le j\le \beta_k\left(\mathbf{d}^m\right)\ . \end{array}\right. \tag{2.10}$$

In fact, the 2nd equality in (2.10) does not contribute much for determination of the degrees of the syzygies, since in accordance with (2.7) the degree of $Q\left(\mathbf{d}^m;z\right)$ is strongly related to the Frobenius number $F\left(\mathbf{d}^m\right)$ which is unknown for $m\ge 3$ in terms of generators d_i only.

If $\beta_1\left(\mathbf{d}^m\right)=m-1$ then the semigroup $\mathsf{S}\left(\mathbf{d}^m\right)$ is called *complete intersection* and the numerator of the corresponding Hilbert series is given by [31]

$$Q\left(\mathbf{d}^m;z\right)=\prod_{j=1}^{m-1}\left(1-z^{e_j}\right)\ ,\quad e_j\in\mathbb{N}\ . \tag{2.11}$$

The degrees of syzygies $C_{j,i}$ provide more precise and accurate characteristics of numerical semigroups than the Betti numbers $\beta_i\left(\mathbf{d}^m\right)$. There are only few sorts of semigroups where both sets of $\beta_i\left(\mathbf{d}^m\right)$ and $C_{j,i}$ are known completely. This is true, e.g., for semigroups of maximal edim ([30], Theorem 1 and [17], sect. 7) and the $\mathsf{S}\left(\mathbf{d}^3\right)$ semigroups of special kind: Pythagorean semigroups ([14], sect. 6.1), pseudosymmetric semigroups ([17], sect. 6.1) and Fibonacci and Lucas symmetric semigroups [16]. A family of semigroups with known Betti numbers but unknown degrees of syzygies is much wider, e.g., nonsymmetric semigroups $\mathsf{S}\left(\mathbf{d}^3\right)$ [19], symmetric semigroups $\mathsf{S}\left(\mathbf{d}^4\right)$ [6], symmetric semigroups of almost maximal edim, $d_1=m+1$ ([30], Theorem 2) and complete intersections [31].

2.1 Main results

In this section we present two theorems on polynomial and quasipolynomial identities for the degrees $C_{j,i}$ of the syzygies. Their proof will follow later, Theorem 1 in section 5.1 and Theorem 2 in section 5.3. Start with a generic case of the generating set $\mathbf{d}^m$ not keeping in mind any relationships among the generators d_j.

Theorem 1 *Let the numerical semigroup* $\mathsf{S}(\mathbf{d}^m)$ *be given with its Hilbert series* $H(\mathbf{d}^m; z)$ *in accordance with (2.2) and (2.3). Then the following polynomial identities hold,*

$$\sum_{j=1}^{\beta_1(\mathbf{d}^m)} C_{j,1}^r - \sum_{j=1}^{\beta_2(\mathbf{d}^m)} C_{j,2}^r + \ldots + (-1)^m \sum_{j=1}^{\beta_{m-1}(\mathbf{d}^m)} C_{j,m-1}^r = 0\,, \qquad r = 1,\ldots,m-2\,, \tag{2.12}$$

$$\sum_{j=1}^{\beta_1(\mathbf{d}^m)} C_{j,1}^{m-1} - \sum_{j=1}^{\beta_2(\mathbf{d}^m)} C_{j,2}^{m-1} + \ldots + (-1)^m \sum_{j=1}^{\beta_{m-1}(\mathbf{d}^m)} C_{j,m-1}^{m-1} = (-1)^m (m-1)! \prod_{i=1}^{m} d_i\,. \tag{2.13}$$

Additional sort of identities appear when we keep in mind relationships among the generators $d_i \in \mathbf{d}^m$. Namely, if there exists a subset $\Xi_q(\mathbf{d}^m) \subset \mathbf{d}^m$ such that $\Xi_q(\mathbf{d}^m) := \{d_i \mid q \mid d_i\}$, $\omega_q = \#\Xi_q(\mathbf{d}^m)$, then there hold another type of identities which are not polynomials.

Theorem 2 *Let the numerical semigroup* $\mathsf{S}(\mathbf{d}^m)$ *be given with its Hilbert series* $H(\mathbf{d}^m; z)$ *in accordance with (2.2) and (2.3). Then for every* $1 < q \leq \max\{d_1,\ldots,d_m\}$, *and* $\gcd(n,q) = 1$, $1 \leq n < q/2$, *the following quasipolynomial identities hold,*

$$\sum_{j=1}^{\beta_1(\mathbf{d}^m)} C_{j,1}^r \exp\left(i\frac{2\pi n}{q} C_{j,1}\right) - \sum_{j=1}^{\beta_2(\mathbf{d}^m)} C_{j,2}^r \exp\left(i\frac{2\pi n}{q} C_{j,2}\right) + \ldots + (-1)^m \sum_{j=1}^{\beta_{m-1}(\mathbf{d}^m)} C_{j,m-1}^r \exp\left(i\frac{2\pi n}{q} C_{j,m-1}\right) = 0$$

where $r = 1,\ldots,\omega_q - 1$. *However, in the case* $r = 0$ *another trigonometric identity holds,*

$$\sum_{j=1}^{\beta_1(\mathbf{d}^m)} \exp\left(i\frac{2\pi n}{q} C_{j,1}\right) - \sum_{j=1}^{\beta_2(\mathbf{d}^m)} \exp\left(i\frac{2\pi n}{q} C_{j,2}\right) + \ldots + (-1)^m \sum_{j=1}^{\beta_{m-1}(\mathbf{d}^m)} \exp\left(i\frac{2\pi n}{q} C_{j,m-1}\right) = 1\,. \tag{2.14}$$

By Theorem 2 another statement comes irrespectively to the inner relationships between the generators d_i. Indeed, by consequence of (2.14) and the fact that the generating set $\mathbf{d}^m$ is minimal we have Corollary.

Corollary 1 *Let the numerical semigroup* $\mathsf{S}(\mathbf{d}^m)$ *be given with its Hilbert series* $H(\mathbf{d}^m; z)$ *in accordance with (2.2) and (2.3). Then for every* $d_k \in \mathbf{d}^m$, $1 \leq k \leq m$, *and* $\gcd(n,d_k) = 1$, $1 \leq n < d_k$, *the following trigonometric identities hold,*

$$\sum_{j=1}^{\beta_1(\mathbf{d}^m)} \exp\left(i\frac{2\pi n}{d_k} C_{j,1}\right) - \sum_{j=1}^{\beta_2(\mathbf{d}^m)} \exp\left(i\frac{2\pi n}{d_k} C_{j,2}\right) + \ldots + (-1)^m \sum_{j=1}^{\beta_{m-1}(\mathbf{d}^m)} \exp\left(i\frac{2\pi n}{d_k} C_{j,m-1}\right) = 1\,. \tag{2.15}$$

In section 6 we will discuss more special cases of numerical semigroups when a part of identities (2.12) and (2.13) becomes trivial (see Corollary 2 for complete intersections) or even they all do not provide new relations (see section 6.1.1 for telescopic semigroups).

3 Restricted Partition Functions $W(s, \mathbf{d}^m)$

The restricted partition function $W(s, \mathbf{d}^m)$ is a number of partitions of s into positive integers $\{d_1, \ldots, d_m\}$, each not greater than s. The generating function $M(\mathbf{d}^m; z)$ for $W(s, \mathbf{d}^m)$ has a form [1],

$$M(\mathbf{d}^m; z) = \prod_{i=1}^{m} \frac{1}{1 - z^{d_i}} = \sum_{s=0}^{\infty} W(s, \mathbf{d}^m)\ z^s\ , \tag{3.1}$$

where $W(s, \mathbf{d}^m)$ satisfies the basic recursive relation

$$W(s, \mathbf{d}^m) - W(s - d_k, \mathbf{d}^m) = W\left(s, \mathbf{d}_k^{m-1}\right), \qquad \mathbf{d}_k^{m-1} = \{d_1, \ldots, d_{k-1}, d_{k+1}, \ldots, d_m\}\ . \tag{3.2}$$

The function $W(s, \mathbf{d}^m)$ is also satisfied parity properties [13], Lemma 4.1,

$$W\left(s - \frac{\sigma_1}{2}, \mathbf{d}^{2m}\right) = -W\left(-s - \frac{\sigma_1}{2}, \mathbf{d}^{2m}\right), \quad W\left(s - \frac{\sigma_1}{2}, \mathbf{d}^{2m+1}\right) = W\left(-s - \frac{\sigma_1}{2}, \mathbf{d}^{2m+1}\right), \tag{3.3}$$

and the statement ([13], Lemma 4.3) about (not all) zeroes $\varsigma_0(\mathbf{d}^m)$ of $W(s, \mathbf{d}^m)$: if the generators d_i are mutually prime numbers, $\gcd(d_i, d_k) = \delta_{ik}$, then

$$\varsigma_0\left(\mathbf{d}^{2m+1}\right) = -1, -2, \ldots, -\sigma_1 + 1 \quad \text{and} \quad \varsigma_0\left(\mathbf{d}^{2m}\right) = -1, -2, \ldots, -\sigma_1 + 1, -\frac{\sigma_1}{2}\ . \tag{3.4}$$

According to definitions of $W(s, \mathbf{d}^m)$ and $F(\mathbf{d}^m)$ of semigroup $\mathsf{S}(\mathbf{d}^m)$, given in section 2, the Frobenius number is a maximal zero of $W(s, \mathbf{d}^m)$.

Regarding the generating function $M(\mathbf{d}^m; z)$ in (3.1), the degree of the numerator in z vanishes ($\deg_z 1 = 0$) while the degree of the denominator in z is positive, $\deg_z \prod_{i=1}^m \left(1 - z^{d_i}\right) = \sigma_1$. Moreover, every zero ζ of the denominator satisfies $\zeta^T = 1$, where $T = lcm(\mathbf{d}^m)$ is a least common multiple of the set $\{d_1, \ldots, d_m\}$. Then by [31], Proposition 4.4.1, the function $W(s, \mathbf{d}^m)$ is a quasipolynomial of degree $m - 1$,

$$W(s, \mathbf{d}^m) = K_1(s, \mathbf{d}^m)\ s^{m-1} + K_2(s, \mathbf{d}^m)\, s^{m-2} + \ldots + K_{m-1}(s, \mathbf{d}^m)\, s + K_m(s, \mathbf{d}^m)\ , \tag{3.5}$$

where each $K_j(s, \mathbf{d}^m)$ is a periodic function with integer period τ_j dividing T. According to Schur's theorem (see [34], Theorem 3.15.2) the 1st coefficient $K_1(s, \mathbf{d}^m)$ is independent of s,

$$K_1(s, \mathbf{d}^m) = \frac{1}{(m-1)!\ \pi_m}\ , \quad \pi_m = \prod_{i=1}^{m} d_i\ . \tag{3.6}$$

As for the other $K_j(s, \mathbf{d}^m)$, $1 < j \le m$, they can be calculated by computational algorithm [13], Appendix A, based on recursion relation (3.2) and zeroes' sequences (3.4).

The partition function $W(s, \mathbf{d}^m)$ for the set of consecutive integers $\{d_1 = 1, \ldots, d_m = m\} = \{\overline{m}\}$ is called *unrestricted*. It was under special consideration in [13] and the corresponding formulas for $W(s, \{\overline{m}\})$, $m \le 12$, were also presented there. Although a variable s in formulas (3.1) – (3.5) is assumed to have integer values, but Rep (3.5) can be extended to real values of s, though

such extension is not unique. It does depend on the choice of complete set of periodic functions for $K_j\left(s,\mathbf{d}^m\right)$ which is often taken as $\mathsf{sin}\left(\frac{2\pi k}{T}s\right)$ and $\mathsf{cos}\left(\frac{2\pi k}{T}s\right)$, $k\in\mathbb{N}$, e.g., [22]

$$
\begin{aligned}
W(s,\{3,5,7\}) = {}& \frac{s^2}{210}+\frac{s}{14}+\frac{74}{315}+\frac{2}{9}\cos\frac{2\pi s}{3}+\frac{8}{25}\left[\left(\sin\frac{\pi}{5}\right)^2\cos\frac{2\pi s}{5}+\left(\sin\frac{2\pi}{5}\right)^2\cos\frac{4\pi s}{5}\right]\\
&-\frac{2}{7\sqrt{7}}\left[\sin\frac{6\pi}{7}\cos\frac{6\pi s}{7}+2\left(\sin\frac{\pi}{7}\right)^2\sin\frac{2\pi s}{7}+2\left(\sin\frac{2\pi}{7}\right)^2\sin\frac{4\pi s}{7}\right]\\
&+\frac{2}{7\sqrt{7}}\left[\sin\frac{2\pi}{7}\cos\frac{2\pi s}{7}+\sin\frac{4\pi}{7}\cos\frac{4\pi s}{7}+2\left(\sin\frac{3\pi}{7}\right)^2\sin\frac{6\pi s}{7}\right]. \quad (3.7)
\end{aligned}
$$

A brief comparison of the last formula with (3.5) shows that $K_2\left(s,\{3,5,7\}\right)$ takes a constant value, this is much stronger than to be periodic in s with period dividing $lcm(3,5,7)=120$. Similar phenomenon holds for other generating sets $\mathbf{d}^m$, $m\geq 3$, with mutually prime generators $\gcd(d_i,d_k)=\delta_{ik}$, i.e., $K_j\left(s,\mathbf{d}^m\right)$, $j=1,\ldots,m-1$, does not depend on s [29]. This indicates that the basic properties (3.2), (3.3) and (3.4) of $W(s,\mathbf{d}^m)$ and general considerations regarding the generating function $M(\mathbf{d}^m;z)$ in (3.1), which were preceding a quasipolynomial Rep (3.5), are quite weak to study $K_j\left(s,\mathbf{d}^m\right)$ in more details.

3.1 Sylvester Waves

A powerful approach to study $W(s,\mathbf{d}^m)$ dates back to Sylvester [33] and his recipe enabling to determine a restricted partition function by decomposing it into *Sylvester waves* $W_q(s,\mathbf{d}^m)$,

$$
W(s,\mathbf{d}^m)=\sum_{q=1,\;q|d_i}^{\max\mathbf{d}^m}W_q(s,\mathbf{d}^m)\;,\quad \max\mathbf{d}^m=\max\{d_1,\ldots,d_m\}\;, \tag{3.8}
$$

where summation runs over all distinct factors of m generators d_i. By (3.8) every wave $W_q(s,\mathbf{d}^m)$ is a quasipolynomial in s and by [29], section 5, it satisfies the recursive relation (3.2).

Sylvester stated and proved [33] that the wave $W_q(s,\mathbf{d}^m)$ is a residue at a point $z=0$ of a function $F_q(s,z)$ which is given by

$$
F_q(s,z)=\sum_{\substack{1\leq n<q\\ \gcd(n,q)=1}}\frac{\xi_q^{-s\,n}\;e^{s\,z}}{\prod_{k=1}^m\left(1-\xi_q^{d_k n}\;e^{-d_k z}\right)}\;,\qquad \xi_q=\exp\left(\frac{2\pi i}{q}\right)\;. \tag{3.9}
$$

The summation in (3.9) is made over all prime roots ξ_q^n for n relatively prime to q (including unity) and smaller than q. Making use of Rep (3.9) Sylvester showed that every wave $W_q(s,\mathbf{d}^m)$ possesses also the parity property (3.3),

$$
W_q\left(s-\frac{\sigma_1}{2},\mathbf{d}^{2m}\right)=-W_q\left(-s-\frac{\sigma_1}{2},\mathbf{d}^{2m}\right),\;W_q\left(s-\frac{\sigma_1}{2},\mathbf{d}^{2m+1}\right)=W_q\left(-s-\frac{\sigma_1}{2},\mathbf{d}^{2m+1}\right) \tag{3.10}
$$

The waves $W_q(s,\mathbf{d}^m)$ were found [29] in a form of finite sum of the Bernoulli polynomials of higher order multiplied by q - periodic function expressed through the Eulerian polynomials of higher

order. In this section we give symbolic formulas for $W_q(s,\mathbf{d}^m)$ which are more appropriate when dealing with higher q [29]. The 1st wave $W_1(s,\mathbf{d}^m)$ is a polynomial part of the whole $W(s,\mathbf{d}^m)$ and serves as a good approximant for the whole $W(s,\mathbf{d}^m)$ [29],

$$W_1(s,\mathbf{d}^m) = \frac{1}{(m-1)!\;\pi_m}\left(s+\sigma_1+\sum_{i=1}^{m} B\; d_i\right)^{m-1}. \tag{3.11}$$

As it is convenient in symbolic (umbral) calculus [26], in (3.11) after binomial expansion the powers $(B\;d_i)^r$ are converted into the generator's powers multiplied by Bernoulli numbers, i.e., $d_i^r B_r$. More details will be given in section 4.1.

The 2nd wave $W_2(s,\mathbf{d}^m)$ reads in symbolic form [29],

$$W_2(s,\mathbf{d}^m) = \frac{2^{\omega_2-m}\cos\pi s}{(\omega_2-1)!\;\pi_{\omega_2}}\left(s+\sigma_1+\sum_{i=1}^{\omega_2} B\; d_i+\sum_{i=\omega_2+1}^{m} E(0)\; d_i\right)^{\omega_2-1}, \tag{3.12}$$

where ω_2 and π_{ω_2} are related to the set $\Xi_2(\mathbf{d}^m)$ comprising only the even generators d_i,

$$\Xi_2(\mathbf{d}^m) := \{d_i \quad | \quad 2\mid d_i\}\,,\quad \omega_2 = \#\Xi_2(\mathbf{d}^m)\,,\quad \pi_{\omega_2} = \prod_{d_i\in\Xi_2(\mathbf{d}^m)} d_i\;.$$

As in formula (3.11), the symbolic binomial expansion the powers $(E(0)\;d_i)^r$ in (3.12) has to be converted into the generator's powers multiplied by the values of the Euler polynomial $E_r(x)$ at $x=0$, i.e., $d_i^r E_r(0)$. Note that $E_r(0)$ differs from the Euler number $E_r = 2^r E_r(1/2)$.

The q-th wave $W_q(s,\mathbf{d}^m)$, $q>1$, reads in symbolic form [29],

$$W_q(s,\mathbf{d}^m) = \frac{1}{(\omega_q-1)!\;\pi_{\omega_q}}\sum_{\substack{1\le n<q\\ \gcd(n,q)=1}} \mathcal{W}_{q,n}(s,\mathbf{d}^m)\,,\qquad \text{where} \tag{3.13}$$

$$\mathcal{W}_{q,n}(s,\mathbf{d}^m) = \frac{\xi_q^{-s\,n}}{\prod_{i=\omega_q+1}^{m}\left(1-\xi_q^{d_i\,n}\right)}\left(s+\sigma_1+\sum_{i=1}^{\omega_q} B\; d_i+\sum_{i=\omega_q+1}^{m} H\left(\xi_q^{d_i\,n}\right)\; d_i\right)^{\omega_q-1}, \tag{3.14}$$

and ω_q and π_{ω_q} are related to the set $\Xi_q(\mathbf{d}^m)$ comprising only the generators d_i divided by q,

$$\Xi_q(\mathbf{d}^m) := \{d_i \quad | \quad q\mid d_i\}\,,\quad \omega_q = \#\Xi_q(\mathbf{d}^m)\,,\quad \pi_{\omega_q} = \prod_{d_i\in\Xi_q(\mathbf{d}^m)} d_i\;.$$

The numbers $\left(H\left(\xi_q^{d_i\,n}\right)\right)^r = H_r\left(\xi_q^{d_i\,n}\right)$ generalize the corresponding $E_r(0) = H_r(-1)$. They were introduced by Frobenius [18] and Carlitz [8] as the values of the rational function $H_n(x)$ at $x=\xi_q^{d_i\,n}$, where $H_n(x)$ itself comes by power expansion of its generating function,

$$\frac{1-x}{e^t-x} = 1+\sum_{n=1}^{\infty} H_n(x)\frac{t^n}{n!},\qquad \text{where}\quad H_n\left(x^{-1}\right)=(-1)^n x\; H_n(x)\,,\quad H_{2n}(-1)=0\,, \tag{3.15}$$

and for $x\neq 1$ the rational function $H_n(x)$ read

$$H_1(x)=\frac{1}{x-1},\quad H_2(x)=\frac{x+1}{(x-1)^2},\quad H_3(x)=\frac{x^2+4x+1}{(x-1)^3},\quad H_4(x)=\frac{x^3+11x^2+11x+1}{(x-1)^4},\;\ldots$$

Consider a special case of the tuple $\mathbf{p}^m = \{p_1, p_2, \ldots, p_m\}$ of primes p_i which leads to essential simplification of formula (3.8). The 1st Sylvester wave is given by (3.11) while all higher waves are purely periodic [29],

$$W_{p_i}(s;\mathbf{p}^m) = \frac{1}{p_i}\sum_{n=1}^{p_i-1}\frac{\xi_{p_i}^{-sn}}{\prod_{r\neq i}^{m}\left(1-\xi_{p_i}^{p_r n}\right)} \ . \tag{3.16}$$

Calculating $W_q(s;\{3,5,7\})$, $q = 1,3,5,7$, one can get (3.7) that explains why $K_2(s,\{3,5,7\})$ in formula (3.5) is taking a constant value. We arrive at the similar conclusion for the generating sets $\mathbf{d}^m$, $m \geq 3$, $\gcd(d_i,d_k) = \delta_{ik}$: since $\omega_q = 1$ in (3.13) then coefficients $K_j(s,\mathbf{d}^m)$, $j = 1,\ldots,m-1$, in (3.5) are taking constant values.

4 Quasipolynomial Representation for $W_q(s,\mathbf{d}^m)$

In this section we specify the quasipolynomial Reps of the Sylvester waves with trigonometric functions as coefficients. Start with technical details and note that by identity $\xi_q^{n_1} = \xi_q^{-n_2}$, $n_1+n_2 = q$, a sum of two partial Sylvester waves $\mathcal{W}_{q,n}(s,\mathbf{d}^m)$ and $\mathcal{W}_{q,q-n}(s,\mathbf{d}^m)$ in (3.13) can be represented as follows

$$\overline{\mathcal{W}}_{q,n}(s,\mathbf{d}^m) = \mathcal{W}_{q,n}(s,\mathbf{d}^m) + \mathcal{W}_{q,-n}(s,\mathbf{d}^m) \ . \tag{4.1}$$

Thus, instead of (3.13) we write

$$W_q(s,\mathbf{d}^m) = \frac{1}{(\omega_q-1)!\ \pi_{\omega_q}}\sum_{\substack{1\leq n<q/2\\ \gcd(n,q)=1}}\overline{\mathcal{W}}_{q,n}(s,\mathbf{d}^m) \ . \tag{4.2}$$

Explicit formulas (3.14), (4.1) and (4.2) for Sylvester waves have one serious lack: their expressions are highly cumbersome and difficult to deal with. On the other hand, a visible simplicity of quasipolynomial (3.5) is accompanied by another lack: periodic functions $K_j(s,\mathbf{d}^m)$ don't distinguish between harmonics with distinct periods. For the purpose of this article it would be worthwhile to have something intermediate, rather simple but still inherited basic properties of (3.13) and (3.14) even if a minor portion of information would left unknown.

Keeping in mind (3.14) choose the following Rep for $\mathcal{W}_{q,\pm n}(s,\mathbf{d}^m)$,

$$\mathcal{W}_{q,\pm n}(s,\mathbf{d}^m) = \frac{\mathsf{L}^{q,n}(s,\mathbf{d}^m) \pm i\ \mathsf{M}^{q,n}(s,\mathbf{d}^m)}{2}\cdot\xi_q^{\mp sn} \ , \tag{4.3}$$

where $\mathsf{L}_m^{q,n}(s)$ and $\mathsf{M}_m^{q,n}(s)$ are real functions. Inserting ξ_q from (3.9) into (4.3) and (4.1) we get

$$\overline{\mathcal{W}}_{q,n}(s,\mathbf{d}^m) = \mathsf{L}^{q,n}(s,\mathbf{d}^m)\cdot\cos\frac{2\pi n}{q}s + \mathsf{M}^{q,n}(s,\mathbf{d}^m)\cdot\sin\frac{2\pi n}{q}s \ . \tag{4.4}$$

Formulas (3.5) and (3.8) for restricted partition function allow to construct one more Rep that reflects the basic properties of the both of them: $W(s,\mathbf{d}^m)$ is a real function comprising the

quasimonomial terms $s^k \sin\left(\frac{2\pi n}{q}s\right)$ and $s^k \cos\left(\frac{2\pi n}{q}s\right)$, $1 \le k \le m$. Thus,

$$W_1(s,\mathbf{d}^m) = \frac{1}{(m-1)!\,\pi_m}\,\mathsf{L}^1(s,\mathbf{d}^m)\,,\quad W_2(s,\mathbf{d}^m) = \frac{2^{\omega_2-m}}{(\omega_2-1)!\,\pi_{\omega_2}}\,\mathsf{L}^{2,1}(s,\mathbf{d}^m)\cdot\cos\pi s\,, \tag{4.5}$$

while $W_q(s,\mathbf{d}^m)$ and $\overline{\mathcal{W}}_{q,n}(s,\mathbf{d}^m)$, $q \ge 3$, are given in (4.2) and (4.4), respectively. Polynomials $\mathsf{L}^1(s,\mathbf{d}^m)$, $\mathsf{L}^{q,n}(s,\mathbf{d}^m)$ and $\mathsf{M}^{q,n}(s,\mathbf{d}^m)$ read

$$\begin{aligned}
\mathsf{L}^1(s,\mathbf{d}^m) &= \ell_1(\mathbf{d}^m)\,s^{m-1} + \ell_2(\mathbf{d}^m)\,s^{m-2} + \ldots + \ell_{m-1}(\mathbf{d}^m)\,s + \ell_m(\mathbf{d}^m)\,, && (4.6)\\
\mathsf{L}^{q,n}(s,\mathbf{d}^m) &= L_1^{q,n}(\mathbf{d}^m)\,s^{\omega_q-1} + L_2^{q,n}(\mathbf{d}^m)\,s^{\omega_q-2} + \ldots + L_{\omega_q}^{q,n}(\mathbf{d}^m)\,,\quad q \ge 2, && (4.7)\\
\mathsf{M}^{q,n}(s,\mathbf{d}^m) &= M_1^{q,n}(\mathbf{d}^m)\,s^{\omega_q-1} + M_2^{q,n}(\mathbf{d}^m)\,s^{\omega_q-2} + \ldots + M_{\omega_q}^{q,n}(\mathbf{d}^m)\,,\quad q \ge 3. && (4.8)
\end{aligned}$$

Coefficients $\ell_j(\mathbf{d}^m)$, $L_j^{q,n}(\mathbf{d}^m)$ and $M_j^{q,n}(\mathbf{d}^m)$ in (4.6) – (4.8) are real and so far unknown.

4.1 The 1st Sylvester wave

Performing a binomial expansion in (3.11) and comparing it with (4.5) we get

$$\mathsf{L}^1(s,\mathbf{d}^m) = \sum_{r=0}^{m-1}\binom{m-1}{r} f_r\cdot s^{m-1-r}\,,\qquad f_r = \left(\sigma_1 + \sum_{i=1}^{m} B\,d_i\right)^r\,, \tag{4.9}$$

where for f_r the above formula presumes a symbolic exponentiation. Denoting $\sigma_k = \sum_{i=1}^m d_i^k$ and making use of the sequence of Bernoulli numbers,

$$B_0 = 1,\ B_1 = -\frac{1}{2},\ B_2 = \frac{1}{6},\ B_4 = -\frac{1}{30},\ B_6 = \frac{1}{42}\,,\ldots,\quad B_{2k+1} = 0\,,\ k \ge 1\,, \tag{4.10}$$

we perform a straightforward calculation in (4.9) and give the seven first values f_r, $0 \le r \le 6$,

$$\begin{aligned}
&f_0 = 1\,,\quad f_1 = \frac{\sigma_1}{2}\,,\quad f_2 = \frac{1}{2^2}\left(\sigma_1^2 - \frac{\sigma_2}{3}\right)\,,\quad f_3 = \frac{\sigma_1}{2^3}\left(\sigma_1^2 - \sigma_2\right)\,, \qquad (4.11)\\
&f_4 = \frac{1}{2^4}\left(\sigma_1^4 - 2\sigma_1^2\sigma_2 + \frac{\sigma_2^2}{3} + \frac{2\sigma_4}{15}\right)\,,\quad f_5 = \frac{\sigma_1}{2^5}\left(\sigma_1^4 - \frac{10\sigma_1^2\sigma_2}{3} + \frac{5\sigma_2^2}{3} + \frac{2\sigma_4}{3}\right)\,,\\
&f_6 = \frac{1}{2^6}\left(\sigma_1^6 - 5\sigma_1^4\sigma_2 + 5\sigma_1^2\sigma_2^2 + 2\sigma_1^2\sigma_4 - \frac{5\sigma_2^3}{9} - \frac{2\sigma_2\sigma_4}{3} - \frac{16\sigma_6}{63}\right)\,.
\end{aligned}$$

In fact, Sylvester's paper [33] contains already formulas for $W_1(s,\mathbf{d}^m)$, $1 \le m \le 7$, where one can recognize f_r given in (4.11). By comparison (4.5), (4.6) and (4.9) we conclude

$$\ell_k(\mathbf{d}^m) = \binom{m-1}{k-1} f_{k-1}\,,\quad \ell_1(\mathbf{d}^m) = 1\,. \tag{4.12}$$

4.2 The 2nd Sylvester wave

Performing a binomial expansion in (3.12) and comparing it with (4.5) we get

$$\mathsf{L}^{2,1}(s,\mathbf{d}^m) = \sum_{r_1,r_2,r_3 \ge 0}^{r_1+r_2+r_3=\omega_2-1} \frac{(\omega_2-1)!}{r_1!\,r_2!\,r_3!}\,l_{r_1}\cdot g_{r_2}\cdot s^{r_3}, \tag{4.13}$$

where for l_r and g_r the above formula presumes a symbolic exponentiation,

$$l_r = \left(\lambda_1 + \sum_{i=1}^{\omega_2} B\ d_i\right)^r, \quad g_r = \left(\gamma_1 + \sum_{i=\omega_2+1}^{m} E(0)\ d_i\right)^r, \quad \lambda_k = \sum_{1\le i\le\omega_2}^{d_i\in\Xi_2(\mathbf{d}^m)} d_i^k, \quad \gamma_k = \sum_{\omega_2<i\le m}^{d_i\notin\Xi_2(\mathbf{d}^m)} d_i^k.$$

Note that $\sigma_k = \lambda_k + \gamma_k$. Making use of sequences of Bernoulli numbers (4.10) and values $E_k(0)$ of Euler polynomials,

$$E_0(0) = 1,\ E_1(0) = -\frac{1}{2},\ E_3(0) = \frac{1}{4},\ E_5(0) = -\frac{1}{2},\ E_7(0) = \frac{17}{8},\ \ldots,\ \ E_{2k} = 0\ ,\ k\ge 1, \quad (4.14)$$

we perform a straightforward calculation in (4.13) and give the six first values g_r, $0\le r\le 5$,

$$g_0 = 1\ , \quad g_1 = \frac{\gamma_1}{2}\ , \quad g_2 = \frac{1}{2^2}\left(\gamma_1^2-\gamma_2\right)\ , \quad g_3 = \frac{\gamma_1}{2^3}\left(\gamma_1^2 - 3\gamma_2\right)\ ,$$
$$g_4 = \frac{1}{2^4}\left(\gamma_1^4 - 6\gamma_1^2\gamma_2 + 3\gamma_2^2 + 2\gamma_4\right)\ , \quad g_5 = \frac{\gamma_1}{2^5}\left(\gamma_1^4 - 10\gamma_1^2\gamma_2 + 15\gamma_2^2 + 10\gamma_4\right)\ .$$

For l_r we have to take corresponding f_r given in (4.11) and replace there σ_k by λ_k, i.e., $l_r(\lambda_1,\lambda_2,\ldots) \to f_r(\sigma_1,\sigma_2,\ldots)$. By comparison (4.5), (4.7) and (4.13) we conclude

$$L_{r+1}^{2,1}(\mathbf{d}^m) = \binom{\omega_2-1}{r}\sum_{k=0}^{r}\binom{r}{k} g_k\cdot l_{r-k}\ , \qquad 0\le r\le\omega_2-1, \quad \text{i.e.,} \qquad (4.15)$$

$$L_1^{2,1}(\mathbf{d}^m) = 1\ , \quad L_2^{2,1}(\mathbf{d}^m) = (\omega_2-1)\frac{\sigma_1}{2}\ , \quad \ldots\ , \quad L_{\omega_2}^{2,1}(\mathbf{d}^m) = \sum_{k=0}^{\omega_2-1}\binom{\omega_2-1}{k} g_k\cdot l_{\omega_2-1-k}\ .$$

4.3 The higher Sylvester waves, $m\ge 3$

Consider the representation (4.3) and rewrite it as follows,

$$\begin{aligned} \mathsf{L}^{q,n}(s,\mathbf{d}^m) &= \mathcal{W}_{q,n}(s,\mathbf{d}^m)\,\xi_q^{sn} + \mathcal{W}_{q,-n}(s,\mathbf{d}^m)\,\xi_q^{-sn}\ ,\\ i\ \mathsf{M}^{q,n}(s,\mathbf{d}^m) &= \mathcal{W}_{q,n}(s,\mathbf{d}^m)\,\xi_q^{sn} - \mathcal{W}_{q,-n}(s,\mathbf{d}^m)\,\xi_q^{-sn}\ . \end{aligned} \qquad (4.16)$$

Substitute a symbolic Rep (3.14) into equalities (4.16) and perform their binomial expansions,

$$\begin{aligned} \mathsf{L}^{q,n}(s,\mathbf{d}^m) &= \sum_{r_1,r_2,r_3\ge 0}^{r_1+r_2+r_3=\omega_q-1} \frac{(\omega_q-1)!}{r_1!\ r_2!\ r_3!}\left(\Pi_{r_2,+}^{q,n}+\Pi_{r_2,-}^{q,n}\right) l_{r_1}^{(q)} s^{r_3}\ ,\\ i\ \mathsf{M}^{q,n}(s,\mathbf{d}^m) &= \sum_{r_1,r_2,r_3\ge 0}^{r_1+r_2+r_3=\omega_q-1} \frac{(\omega_q-1)!}{r_1!\ r_2!\ r_3!}\left(\Pi_{r_2,+}^{q,n}-\Pi_{r_2,-}^{q,n}\right) l_{r_1}^{(q)} s^{r_3}\ , \end{aligned} \qquad (4.17)$$

where

$$\Pi_{r,\pm}^{q,n} = \frac{h_{r,\pm}^{q,n}}{\prod_{i=\omega_q+1}^{m}\left(1-\xi_q^{\pm d_i\ n}\right)}\ , \qquad h_{r,\pm}^{q,n} = \left(\gamma_1^{(q)} + \sum_{i=\omega_q+1}^{m} H\left(\xi_q^{\pm d_i\ n}\right)\ d_i\right)^r\ , \qquad (4.18)$$

$$l_r^{(q)} = \left(\lambda_1^{(q)} + \sum_{i=1}^{\omega_q} B\ d_i\right)^r, \quad \lambda_k^{(q)} = \sum_{1\le i\le\omega_q}^{d_i\in\Xi_q(\mathbf{d}^m)} d_i^k, \quad \gamma_k^{(q)} = \sum_{\omega_q<i\le m}^{d_i\notin\Xi_q(\mathbf{d}^m)} d_i^k, \quad \lambda_k^{(q)}+\gamma_k^{(q)} = \sigma_k.$$

By comparison (4.5), (4.7), (4.8) and (4.17) we conclude

$$L_{r+1}^{q,n}\left(\mathbf{d}^m\right) = \binom{\omega_q-1}{r}\sum_{k=0}^{r}\binom{r}{k}\left(\Pi_{k,+}^{q,n}+\Pi_{k,-}^{q,n}\right)\cdot l_{r-k}^{(q)}\;, \qquad 0\le r\le \omega_q-1, \tag{4.19}$$

$$i\,M_{r+1}^{q,n}\left(\mathbf{d}^m\right) = \binom{\omega_q-1}{r}\sum_{k=0}^{r}\binom{r}{k}\left(\Pi_{k,+}^{q,n}-\Pi_{k,-}^{q,n}\right)\cdot l_{r-k}^{(q)}\;. \tag{4.20}$$

It is easy to calculate the 1st pair of coefficients

$$L_1^{q,n}\left(\mathbf{d}^m\right)=\frac{1+(-1)^{m-\omega_q}\xi_q^{\sigma_1\,n}}{\prod_{i=\omega_q+1}^m\left(1-\xi_q^{d_i\,n}\right)}\;,\qquad i\,M_1^{q,n}\left(\mathbf{d}^m\right)=\frac{1-(-1)^{m-\omega_q}\xi_q^{\sigma_1\,n}}{\prod_{i=\omega_q+1}^m\left(1-\xi_q^{d_i\,n}\right)}\;, \tag{4.21}$$

and note that both numbers $L_1^{q,n}\left(\mathbf{d}^m\right)$ and $M_1^{q,n}\left(\mathbf{d}^m\right)$ cannot vanish simultaneously. Lemma 1 provide selection rules for these coefficients and explains why some of them disappear in (3.7).

Lemma 1

1. $$\frac{\sigma_1\cdot n}{q}=k,\quad k\in\mathbb{N}\;, \tag{4.22}$$

$$\mathsf{L}_1^{q,n}\left(\mathbf{d}^m\right)=0,\quad\text{if}\quad\begin{cases}2\mid m,\;2\nmid\omega_q,\\ 2\nmid m,\;2\mid\omega_q,\end{cases}\quad\text{and}\quad \mathsf{M}_1^{q,n}\left(\mathbf{d}^m\right)=0,\quad\text{if}\quad\begin{cases}2\mid m,\;2\mid\omega_q,\\ 2\nmid m,\;2\nmid\omega_q.\end{cases}$$

2. $$\frac{\sigma_1\cdot n}{q}=k+\frac{1}{2},\quad k\in\mathbb{N}\;, \tag{4.23}$$

$$\mathsf{M}_1^{q,n}\left(\mathbf{d}^m\right)=0,\quad\text{if}\quad\begin{cases}2\mid m,\;2\nmid\omega_q,\\ 2\nmid m,\;2\mid\omega_q,\end{cases}\quad\text{and}\quad \mathsf{L}_1^{q,n}\left(\mathbf{d}^m\right)=0,\quad\text{if}\quad\begin{cases}2\mid m,\;2\mid\omega_q,\\ 2\nmid m,\;2\nmid\omega_q.\end{cases}$$

5 Identities for Degrees of the Syzygies

In this section we bring together two different approaches, theory of restricted partition and commutative semigroup rings theory, and prove the main Theorems 1 and 2. We start with simple identity which comes by comparison of (2.2) and (3.1),

$$\sum_{s\,\in\,\mathsf{S}(\mathbf{d}^m)} z^s = Q\left(\mathbf{d}^m;z\right)\sum_{s\,\in\,\mathsf{S}(\mathbf{d}^m)} W\left(\mathbf{d}^m;s\right)z^s\;. \tag{5.1}$$

Substituting (2.3) and (2.4) into (5.1) we get

$$\begin{aligned}\sum_{s\,\in\,\mathsf{S}(\mathbf{d}^m)} z^s \;=\;& \sum_{s\,\in\,\mathsf{S}(\mathbf{d}^m)} W\left(\mathbf{d}^m;s\right)z^s-\sum_{j=1}^{\beta_1(\mathbf{d}^m)}\sum_{s\,\in\,\mathsf{S}(\mathbf{d}^m)} W\left(\mathbf{d}^m;s\right)z^{C_{j,1}+s}+\ldots+\\ &(-1)^{m-1}\sum_{j=1}^{\beta_{m-1}(\mathbf{d}^m)}\sum_{s\,\in\,\mathsf{S}(\mathbf{d}^m)} W\left(\mathbf{d}^m;s\right)z^{C_{j,m-1}+s}\;.\end{aligned} \tag{5.2}$$

Rewrite an equality (5.2) as follows

$$\begin{aligned}\sum_{s\,\in\,\mathsf{S}(\mathbf{d}^m)} z^s \;=\;& \sum_{s\,\in\,\mathsf{S}(\mathbf{d}^m)} W\left(\mathbf{d}^m;s\right)z^s-\sum_{j=1}^{\beta_1(\mathbf{d}^m)}\sum_{s\,\in\,\mathsf{S}(\mathbf{d}^m)} W\left(\mathbf{d}^m;s-C_{j,1}\right)z^s+\ldots+\\ &(-1)^{m-1}\sum_{j=1}^{\beta_{m-1}(\mathbf{d}^m)}\sum_{s\,\in\,\mathsf{S}(\mathbf{d}^m)} W\left(\mathbf{d}^m;s-C_{j,m-1}\right)z^s\;,\end{aligned} \tag{5.3}$$

and equate the corresponding contributions coming from monomial terms z^s in the left hand side (l.h.s.) and right hand side (r.h.s.) of (5.3). This gives a quasipolynomial equality,

$$W\left(\mathbf{d}^m;s\right)-\sum_{j=1}^{\beta_1(\mathbf{d}^m)} W\left(\mathbf{d}^m;s-C_{j,1}\right)+\ldots+(-1)^{m-1}\sum_{j=1}^{\beta_{m-1}(\mathbf{d}^m)} W\left(\mathbf{d}^m;s-C_{j,m-1}\right)=1\ . \quad (5.4)$$

Now substitute into (5.4) the Sylvester expansions (3.8), (4.2) and make use of the linear independence of the partial Sylvester waves $W_q(s,\mathbf{d}^m)$ and $\overline{\mathcal{W}}_{q,n}(s,\mathbf{d}^m)$.

This gives rise to a set of quasipolynomial equalities,

$$W_1(s,\mathbf{d}^m)-\sum_{j=1}^{\beta_1(\mathbf{d}^m)} W_1(s-C_{j,1},\mathbf{d}^m)+\ldots+(-1)^{m-1}\sum_{j=1}^{\beta_{m-1}(\mathbf{d}^m)} W_1(s-C_{j,m-1},\mathbf{d}^m)=1, \quad (5.5)$$

and for $q\geq 2$, $1\leq n<q/2$ such that $\gcd(n,q)=1$,

$$\overline{\mathcal{W}}_{q,n}(s,\mathbf{d}^m)-\sum_{j=1}^{\beta_1(\mathbf{d}^m)} \overline{\mathcal{W}}_{q,n}(s-C_{j,1},\mathbf{d}^m)+\ldots+(-1)^{m-1}\sum_{j=1}^{\beta_{m-1}(\mathbf{d}^m)} \overline{\mathcal{W}}_{q,n}\left(s-C_{j,m-1},\mathbf{d}^m\right)=0. \quad (5.6)$$

Equalities (5.5) and (5.6) are a source of new relationships between degrees $C_{j,i}$ of the syzygies of different kinds. What is more important that for this purpose we have to know only the most basic properties of partial Sylvester waves $W_q(s,\mathbf{d}^m)$ such as their quasipolynomial Reps (4.2), (4.4) – (4.8) and the coefficients $\ell_1\left(\mathbf{d}^m\right)$, $L_1^{q,n}\left(\mathbf{d}^m\right)$ and $M_1^{q,n}\left(\mathbf{d}^m\right)$ at the leading terms (4.12), (4.15) and (4.21) but not the whole set of quasipolynomial coefficients in (4.6) – (4.8).

5.1 Polynomial Identities Associated with the 1st Sylvester Wave. The Proof of Theorem 1

In this section we prove Theorem 1 which was stated in section 2.1. Consider equality (5.5) and substitute Rep (4.5) for $W_1(s,\mathbf{d}^m)$ into (5.5),

$$\mathsf{L}^1\left(s,\mathbf{d}^m\right)-\sum_{j=1}^{\beta_1(\mathbf{d}^m)} \mathsf{L}^1(s-C_{j,1},\mathbf{d}^m)+\ldots+(-1)^{m-1}\sum_{j=1}^{\beta_{m-1}(\mathbf{d}^m)} \mathsf{L}^1(s-C_{j,m-1},\mathbf{d}^m)=(m-1)!\ \pi_m.$$

Inserting Rep (4.6) into the last identity we get

$$\begin{aligned}
&\ell_1\left(\mathbf{d}^m\right)\left[s^{m-1}-\sum_{j=1}^{\beta_1(\mathbf{d}^m)}\left(s-C_{j,1}\right)^{m-1}+\ldots+(-1)^{m-1}\sum_{j=1}^{\beta_{m-1}(\mathbf{d}^m)}\left(s-C_{j,m-1}\right)^{m-1}\right]+\\
&\ell_2\left(\mathbf{d}^m\right)\left[s^{m-2}-\sum_{j=1}^{\beta_1(\mathbf{d}^m)}\left(s-C_{j,1}\right)^{m-2}+\ldots+(-1)^{m-1}\sum_{j=1}^{\beta_{m-2}(\mathbf{d}^m)}\left(s-C_{j,m-1}\right)^{m-2}\right]+\ldots+\\
&\ell_{m-1}\left(\mathbf{d}^m\right)\left[s-\sum_{j=1}^{\beta_1(\mathbf{d}^m)}\left(s-C_{j,1}\right)+\ldots+(-1)^{m-1}\sum_{j=1}^{\beta_{m-1}(\mathbf{d}^m)}\left(s-C_{j,m-1}\right)\right]+\\
&\ell_m\left(\mathbf{d}^m\right)\left[1-\sum_{j=1}^{\beta_1(\mathbf{d}^m)}1+\ldots+(-1)^{m-1}\sum_{j=1}^{\beta_{m-1}(\mathbf{d}^m)}1\right]=(m-1)!\ \pi_m\ .
\end{aligned} \quad (5.7)$$

Introduce the following sums and for short denote them as follows,

$$\begin{aligned}
\mathbf{A}_0\left(\mathbf{d}^m\right) &= 1-\beta_1\left(\mathbf{d}^m\right)+\beta_2\left(\mathbf{d}^m\right)-\ldots+(-1)^{m-1}\beta_{m-1}\left(\mathbf{d}^m\right)\ , \\
\mathbf{A}_k\left(\mathbf{d}^m\right) &= \sum_{j=1}^{\beta_1(\mathbf{d}^m)} C_{j,1}^k-\sum_{j=1}^{\beta_2(\mathbf{d}^m)} C_{j,2}^k+\ldots+(-1)^m\sum_{j=1}^{\beta_{m-1}(\mathbf{d}^m)} C_{j,m-1}^k\ .
\end{aligned} \tag{5.8}$$

Keeping in mind (5.8) rewrite equality (5.7) and get

$$\begin{aligned}
&\left[\ell_1\left(\mathbf{d}^m\right)s^{m-1}+\ell_2\left(\mathbf{d}^m\right)s^{m-2}+\ldots+\ell_m\left(\mathbf{d}^m\right)\right]\mathbf{A}_0\left(\mathbf{d}^m\right)+ \\
&\left[\ell_1\left(\mathbf{d}^m\right)\binom{m-1}{1}s^{m-2}+\ell_2\left(\mathbf{d}^m\right)\binom{m-2}{1}s^{m-3}+\ldots+\ell_{m-1}\left(\mathbf{d}^m\right)\binom{1}{1}\right]\mathbf{A}_1\left(\mathbf{d}^m\right)- \\
&\left[\ell_1\left(\mathbf{d}^m\right)\binom{m-1}{2}s^{m-3}+\ell_2\left(\mathbf{d}^m\right)\binom{m-2}{2}s^{m-4}+\ldots+\ell_{m-2}\left(\mathbf{d}^m\right)\binom{2}{2}\right]\mathbf{A}_2\left(\mathbf{d}^m\right)+\ldots- \\
&(-1)^m\left[\ell_1\left(\mathbf{d}^m\right)\binom{m-1}{m-2}s+\ell_2\left(\mathbf{d}^m\right)\right]\mathbf{A}_{m-2}\left(\mathbf{d}^m\right)+ \\
&(-1)^m\,\ell_1\left(\mathbf{d}^m\right)\mathbf{A}_{m-1}\left(\mathbf{d}^m\right)=(m-1)!\pi_m\ .
\end{aligned} \tag{5.9}$$

Equating the corresponding contributions coming from the power terms s^a, $0\le a<m$, in the l.h.s. and the r.h.s. of (5.9) and keeping in mind $\ell_1\left(\mathbf{d}^m\right)=1$ (see (4.12)), we get finally,

$$\mathbf{A}_k\left(\mathbf{d}^m\right)=0\ ,\quad k=0,1,\ldots,m-2\ ,\qquad \mathbf{A}_{m-1}\left(\mathbf{d}^m\right)=(-1)^m(m-1)!\ \pi_m\ . \tag{5.10}$$

The 1st identity $\mathbf{A}_0\left(\mathbf{d}^m\right)=0$ is already known in (2.5). The rest of identities prove Theorem 1.

5.2 Quasipolynomial Identities Associated with the 2nd Sylvester Wave

In this section we study an intermediate case $q=2$ of the master equality (5.6) which is technically slightly more difficult than the previous equality (5.5). Consider (5.6) and substitute Rep (4.5) for $W_2(s,\mathbf{d}^m)=\overline{\mathcal{W}}_{2,1}\left(s,\mathbf{d}^m\right)$ into (5.6),

$$\begin{aligned}
&\mathsf{L}^{2,1}\left(s,\mathbf{d}^m\right)\cos\pi s-\sum_{j=1}^{\beta_1(\mathbf{d}^m)}\mathsf{L}^{2,1}(s-C_{j,1},\mathbf{d}^m)\cos\pi\left(s-C_{j,1}\right)+\ldots+ \\
&\qquad (-1)^{m-1}\sum_{j=1}^{\beta_{m-1}(\mathbf{d}^m)}\mathsf{L}^{2,1}(s-C_{j,m-1},\mathbf{d}^m)\cos\pi\left(s-C_{j,m-1}\right)=0\ .
\end{aligned} \tag{5.11}$$

Substituting Rep (4.7) into the last equality we get

$$\begin{aligned}
&\left(L_1^{2,1}\left(\mathbf{d}^m\right)s^{\omega_2-1}+L_2^{2,1}\left(\mathbf{d}^m\right)s^{\omega_2-2}+\ldots+L_{\omega_2-1}^{2,1}\left(\mathbf{d}^m\right)s+L_{\omega_2}^{2,1}\left(\mathbf{d}^m\right)\right)\cos\pi s- \\
&\qquad\sum_{j=1}^{\beta_1(\mathbf{d}^m)}\left(L_1^{2,1}\left(\mathbf{d}^m\right)\left(s-C_{j,1}\right)^{\omega_2-1}+\ldots+L_{\omega_2}^{2,1}\left(\mathbf{d}^m\right)\right)\cos\pi\left(s-C_{j,1}\right)+ \\
&\ldots+(-1)^{m-1}\sum_{j=1}^{\beta_{m-1}(\mathbf{d}^m)}\left(L_1^{2,1}\left(\mathbf{d}^m\right)\left(s-C_{j,m-1}\right)^{\omega_2-1}+\ldots+L_{\omega_2}^{2,1}\left(\mathbf{d}^m\right)\right)\cos\pi\left(s-C_{j,m-1}\right)=0\ .
\end{aligned}$$

Keeping in mind two identities $\sin\pi s=0$ and $\cos\pi s\neq 0$, $s\in\mathbb{N}$, we write,

$$
\begin{aligned}
&\left(L_1^{2,1}(\mathbf{d}^m)\,s^{\omega_2-1}+L_2^{2,1}(\mathbf{d}^m)\,s^{\omega_2-2}+\ldots+L_{\omega_2-1}^{2,1}(\mathbf{d}^m)\,s+L_{\omega_2}^{2,1}(\mathbf{d}^m)\right)- \\
&\sum_{j=1}^{\beta_1(\mathbf{d}^m)}\left(L_1^{2,1}(\mathbf{d}^m)\,(s-C_{j,1})^{\omega_2-1}+\ldots+L_{\omega_2}^{2,1}(\mathbf{d}^m)\right)\cos(\pi C_{j,1})+\ldots+ \\
&(-1)^{m-1}\sum_{j=1}^{\beta_{m-1}(\mathbf{d}^m)}\left(L_1^{2,1}(\mathbf{d}^m)\,(s-C_{j,m-1})^{\omega_2-1}+\ldots+L_{\omega_2}^{2,1}(\mathbf{d}^m)\right)\cos(\pi C_{j,m-1})=0.
\end{aligned}
\tag{5.12}
$$

Rearrange the last equality (5.12) as follows,

$$
\begin{aligned}
&L_1^{2,1}(\mathbf{d}^m)\left[s^{\omega_2-1}-\sum_{j=1}^{\beta_1(\mathbf{d}^m)}(s-C_{j,1})^{\omega_2-1}\cos(\pi C_{j,1})+\ldots+\right. \\
&\qquad\left.(-1)^{m-1}\sum_{j=1}^{\beta_{m-1}(\mathbf{d}^m)}(s-C_{j,m-1})^{\omega_2-1}\cos(\pi C_{j,m-1})\right]+ \\
&L_2^{2,1}(\mathbf{d}^m)\left[s^{\omega_2-2}-\sum_{j=1}^{\beta_1(\mathbf{d}^m)}(s-C_{j,1})^{\omega_2-2}\cos(\pi C_{j,1})+\ldots+\right. \\
&\qquad\left.(-1)^{m-1}\sum_{j=1}^{\beta_{m-2}(\mathbf{d}^m)}(s-C_{j,m-1})^{\omega_2-2}\cos(\pi C_{j,m-1})\right]+\ldots+ \\
&L_{\omega_2}^{2,1}(\mathbf{d}^m)\left[1-\sum_{j=1}^{\beta_1(\mathbf{d}^m)}\cos(\pi C_{j,1})+\ldots+(-1)^{m-1}\sum_{j=1}^{\beta_{m-1}(\mathbf{d}^m)}\cos(\pi C_{j,m-1})\right]=0\,.
\end{aligned}
\tag{5.13}
$$

and denote two following sums,

$$
\begin{aligned}
\mathbf{B}_0(\mathbf{d}^m) &= 1-\sum_{j=1}^{\beta_1(\mathbf{d}^m)}\cos(\pi C_{j,1})+\ldots+(-1)^{m-1}\sum_{j=1}^{\beta_{m-1}(\mathbf{d}^m)}\cos(\pi C_{j,m-1})\ , \\
\mathbf{B}_k(\mathbf{d}^m) &= \sum_{j=1}^{\beta_1(\mathbf{d}^m)}C_{j,1}^k\cos(\pi C_{j,1})-\ldots+(-1)^m\sum_{j=1}^{\beta_{m-1}(\mathbf{d}^m)}C_{j,m-1}^k\cos(\pi C_{j,m-1})\,.
\end{aligned}
\tag{5.14}
$$

Keeping in mind (5.14) rewrite equality (5.13) and get

$$
\begin{aligned}
&\left[L_1^{2,1}(\mathbf{d}^m)\,s^{\omega_2-1}+L_2^{2,1}(\mathbf{d}^m)\,s^{\omega_2-2}+\ldots+L_{\omega_2}^{2,1}(\mathbf{d}^m)\right]\mathbf{B}_0(\mathbf{d}^m)+ \\
&\left[L_1^{2,1}(\mathbf{d}^m)\binom{\omega_2-1}{1}s^{\omega_2-2}+L_2^{2,1}(\mathbf{d}^m)\binom{\omega_2-2}{1}s^{\omega_2-3}+\ldots+L_{\omega_2-1}^{2,1}(\mathbf{d}^m)\right]\mathbf{B}_1(\mathbf{d}^m)- \\
&\left[L_1^{2,1}(\mathbf{d}^m)\binom{\omega_2-1}{2}s^{\omega_2-3}+L_2^{2,1}(\mathbf{d}^m)\binom{\omega_2-2}{2}s^{\omega_2-4}+\ldots+L_{\omega_2-2}^{2,1}(\mathbf{d}^m)\right]\mathbf{B}_2(\mathbf{d}^m)+\ldots- \\
&(-1)^{\omega_2}\left[L_1^{2,1}(\mathbf{d}^m)\binom{\omega_2-1}{\omega_2-2}s+L_2^{2,1}(\mathbf{d}^m)\right]\mathbf{B}_{\omega_2-2}(\mathbf{d}^m)+ \\
&(-1)^{\omega_2}L_1^{2,1}(\mathbf{d}^m)\,\mathbf{B}_{\omega_2-1}(\mathbf{d}^m)=0\,.
\end{aligned}
\tag{5.15}
$$

Equating the corresponding contributions coming from the power terms s^a, $0\le a<\omega_2$, in the l.h.s. and the r.h.s. of (5.15) and keeping in mind $L_1^{2,1}(\mathbf{d}^m)=1$ (see (4.15)), we get finally,

$$
\mathbf{B}_k(\mathbf{d}^m)=0\ ,\quad k=0,1,\ldots,\omega_2-1\,. \tag{5.16}
$$

5.3 Quasipolynomial Identities Associated with the Higher Sylvester Waves. The Proof of Theorem 2

In this section we study the general case of equality (5.6) and start to prove Theorem 2, we finish its proof in section 5.3.3. Consider (5.6) and substitute Reps (4.2) and (4.4) into (5.6),

$$\mathsf{L}^{q,n}\left(s,\mathbf{d}^m\right)\cos\frac{2\pi n}{q}s+\mathsf{M}^{q,n}\left(s,\mathbf{d}^m\right)\sin\frac{2\pi n}{q}s-$$
$$\sum_{j=1}^{\beta_1(\mathbf{d}^m)}\left(\mathsf{L}^{q,n}(s-C_{j,1},\mathbf{d}^m)\cos\frac{2\pi n}{q}\left(s-C_{j,1}\right)+\mathsf{M}^{q,n}(s-C_{j,1},\mathbf{d}^m)\sin\frac{2\pi n}{q}\left(s-C_{j,1}\right)\right)+$$
$$\sum_{j=1}^{\beta_2(\mathbf{d}^m)}\left(\mathsf{L}^{q,n}(s-C_{j,2},\mathbf{d}^m)\cos\frac{2\pi n}{q}\left(s-C_{j,2}\right)+\mathsf{M}^{q,n}(s-C_{j,2},\mathbf{d}^m)\sin\frac{2\pi n}{q}\left(s-C_{j,2}\right)\right)-$$
$$\ldots+(-1)^{m-1}\sum_{j=1}^{\beta_{m-1}(\mathbf{d}^m)}\left(\mathsf{L}^{q,n}(s-C_{j,m-1},\mathbf{d}^m)\cos\frac{2\pi n}{q}\left(s-C_{j,m-1}\right)+\right.$$
$$\left.\mathsf{M}^{q,n}(s-C_{j,1},\mathbf{d}^m)\sin\frac{2\pi n}{q}\left(s-C_{j,1},\mathbf{d}^m\right)\right)=0\;.$$

For $q\geq 2$, $1\leq n<q/2$, $\gcd(n,q)=1$, we represent the last equality in the form

$$\mathbb{A}^{q,n}\left(s,\mathbf{d}^m\right)\cos\frac{2\pi n}{q}s+\mathbb{B}^{q,n}\left(s,\mathbf{d}^m\right)\sin\frac{2\pi n}{q}s=0\;,\tag{5.17}$$

where $\mathbb{A}^{q,n}\left(s,\mathbf{d}^m\right)$ and $\mathbb{B}^{q,n}\left(s,\mathbf{d}^m\right)$ are two polynomials in s which can be calculated,

$$\mathbb{A}^{q,n}\left(s,\mathbf{d}^m\right)=\mathsf{L}^{q,n}\left(s,\mathbf{d}^m\right)-\sum_{j=1}^{\beta_1(\mathbf{d}^m)}\mathsf{L}^{q,n}(s-C_{j,1},\mathbf{d}^m)\cos\frac{2\pi n}{q}C_{j,1}+$$
$$\sum_{j=1}^{\beta_2(\mathbf{d}^m)}\mathsf{L}^{q,n}(s-C_{j,2},\mathbf{d}^m)\cos\frac{2\pi n}{q}C_{j,2}-\ldots+(-1)^{m-1}\sum_{j=1}^{\beta_{m-1}(\mathbf{d}^m)}\mathsf{L}^{q,n}(s-C_{j,m-1},\mathbf{d}^m)\cos\frac{2\pi n}{q}C_{j,m-1}+$$
$$\sum_{j=1}^{\beta_1(\mathbf{d}^m)}\mathsf{M}^{q,n}(s-C_{j,1},\mathbf{d}^m)\sin\frac{2\pi n}{q}C_{j,1}-\ldots-(-1)^{m-1}\sum_{j=1}^{\beta_{m-1}(\mathbf{d}^m)}\mathsf{M}^{q,n}\left(s-C_{j,m-1},\mathbf{d}^m\right)\sin\frac{2\pi n}{q}C_{j,m-1}$$

$$\mathbb{B}^{q,n}\left(s,\mathbf{d}^m\right)=\mathsf{M}^{q,n}\left(s,\mathbf{d}^m\right)-\sum_{j=1}^{\beta_1(\mathbf{d}^m)}\mathsf{M}^{q,n}(s-C_{j,1},\mathbf{d}^m)\cos\frac{2\pi n}{q}C_{j,1}+$$
$$\sum_{j=1}^{\beta_2(\mathbf{d}^m)}\mathsf{M}^{q,n}(s-C_{j,2},\mathbf{d}^m)\cos\frac{2\pi n}{q}C_{j,2}-\ldots+(-1)^{m-1}\sum_{j=1}^{\beta_{m-1}(\mathbf{d}^m)}\mathsf{M}^{q,n}(s-C_{j,m-1},\mathbf{d}^m)\cos\frac{2\pi n}{q}C_{j,m-1}-$$
$$\sum_{j=1}^{\beta_1(\mathbf{d}^m)}\mathsf{L}^{q,n}(s-C_{j,1},\mathbf{d}^m)\sin\frac{2\pi n}{q}C_{j,1}+\ldots+(-1)^{m-1}\sum_{j=1}^{\beta_{m-1}(\mathbf{d}^m)}\mathsf{L}^{q,n}\left(s-C_{j,m-1},\mathbf{d}^m\right)\sin\frac{2\pi n}{q}C_{j,m-1}$$

The quasipolynomial in the l.h.s. of (5.17) vanishes identically iff all coefficients at every harmonics $\cos\frac{2\pi n}{q}s$ and $\sin\frac{2\pi n}{q}s$ do vanish. This is why (5.17) is equivalent two independent Eqs,

$$\mathbb{A}^{q,n}\left(s,\mathbf{d}^m\right)=0\;,\qquad\mathbb{B}^{q,n}\left(s,\mathbf{d}^m\right)=0\;,\tag{5.18}$$

which have to be satisfied identically but solved separately.

5.3.1 $\mathbb{A}^{q,n}(s,\mathbf{d}^m)=0$

In this section we find the 1st system of linear equations (5.26) for the coefficients $L_k^{q,n}(\mathbf{d}^m)$ and $M_k^{q,n}(\mathbf{d}^m)$, $1\le k\le\omega_q$. First, in order to avoid the lengthy formulas coming from (5.18) we introduce another pair of quasipolynomials equipped with subscript p, $0\le p<\omega_q$,

$$\begin{aligned}\mathbb{C}_p^{q,n}(s,\mathbf{d}^m) &= \sum_{j=1}^{\beta_1(\mathbf{d}^m)}(s-C_{j,1})^p\cos\frac{2\pi n}{q}C_{j,1}-\sum_{j=1}^{\beta_2(\mathbf{d}^m)}(s-C_{j,2})^p\cos\frac{2\pi n}{q}C_{j,2}+\ldots-\\ &\qquad (-1)^{m-1}\sum_{j=1}^{\beta_{m-1}(\mathbf{d}^m)}(s-C_{j,m-1})^p\cos\frac{2\pi n}{q}C_{j,m-1}\,,\end{aligned}\tag{5.19}$$

$$\begin{aligned}\mathbb{S}_p^{q,n}(s,\mathbf{d}^m) &= \sum_{j=1}^{\beta_1(\mathbf{d}^m)}(s-C_{j,1})^p\sin\frac{2\pi n}{q}C_{j,1}-\sum_{j=1}^{\beta_2(\mathbf{d}^m)}(s-C_{j,2})^p\sin\frac{2\pi n}{q}C_{j,2}+\ldots-\\ &\qquad (-1)^{m-1}\sum_{j=1}^{\beta_{m-1}(\mathbf{d}^m)}(s-C_{j,m-1})^p\sin\frac{2\pi n}{q}C_{j,m-1}\,.\end{aligned}\tag{5.20}$$

Substitute now Reps (4.7) and (4.8) into expression for $\mathbb{A}^{q,n}(s,\mathbf{d}^m)$ given in the previous section and make use of (5.19) and (5.20). Thus, we present the 1st Eq. in (5.18) as follows,

$$\begin{aligned}&L_1^{q,n}(\mathbf{d}^m)\left[s^{\omega_q-1}-\mathbb{C}_{\omega_q-1}^{q,n}(s,\mathbf{d}^m)\right]+L_2^{q,n}(\mathbf{d}^m)\left[s^{\omega_q-2}-\mathbb{C}_{\omega_q-2}^{q,n}(s,\mathbf{d}^m)\right]+\ldots+\\ &L_{\omega_q-1}^{q,n}(\mathbf{d}^m)\left[s-\mathbb{C}_1^{q,n}(s,\mathbf{d}^m)\right]+L_{\omega_q}^{q,n}(\mathbf{d}^m)\left[1-\mathbb{C}_0^{q,n}(s,\mathbf{d}^m)\right]+\\ &M_1^{q,n}(\mathbf{d}^m)\,\mathbb{S}_{\omega_q-1}^{q,n}(s,\mathbf{d}^m)+M_2^{q,n}(\mathbf{d}^m)\,\mathbb{S}_{\omega_q-2}^{q,n}(s,\mathbf{d}^m)+\ldots+M_{\omega_q}^{q,n}(\mathbf{d}^m)\,\mathbb{S}_0^{q,n}(s,\mathbf{d}^m)=0\,.\end{aligned}\tag{5.21}$$

After binomial expansion of the terms $(s-C_{j,i})^p$ in (5.19) and (5.20) and subsequent substitution into (5.21) we recast the power terms in s and get the final equality

$$\begin{aligned}&\left[L_1^{q,n}(\mathbf{d}^m)\,s^{\omega_q-1}+L_2^{q,n}(\mathbf{d}^m)\,s^{\omega_q-2}+\ldots+L_{\omega_q}^{q,n}(\mathbf{d}^m)\right]\mathbf{G}_0(\mathbf{d}^m)+\\ &\left[L_1^{q,n}(\mathbf{d}^m)\binom{\omega_q-1}{1}s^{\omega_q-2}+L_2^{q,n}(\mathbf{d}^m)\binom{\omega_q-2}{1}s^{\omega_q-3}+\ldots+L_{\omega_q-1}^{q,n}(\mathbf{d}^m)\right]\mathbf{G}_1(\mathbf{d}^m)-\\ &\left[L_1^{q,n}(\mathbf{d}^m)\binom{\omega_q-1}{2}s^{\omega_q-3}+L_2^{q,n}(\mathbf{d}^m)\binom{\omega_q-2}{2}s^{\omega_q-4}+\ldots+L_{\omega_q-2}^{q,n}(\mathbf{d}^m)\right]\mathbf{G}_2(\mathbf{d}^m)+\ldots-\\ &(-1)^{\omega_q}\left[L_1^{q,n}(\mathbf{d}^m)\binom{\omega_q-1}{\omega_q-2}s+L_2^{q,n}(\mathbf{d}^m)\right]\mathbf{G}_{\omega_q-2}(\mathbf{d}^m)+(-1)^{\omega_q}\,L_1^{q,n}(\mathbf{d}^m)\,\mathbf{G}_{\omega_q-1}(\mathbf{d}^m)+\\ &\left[M_1^{q,n}(\mathbf{d}^m)\,s^{\omega_q-1}+M_2^{q,n}(\mathbf{d}^m)\,s^{\omega_q-2}+\ldots+M_{\omega_q}^{q,n}(\mathbf{d}^m)\right]\mathbf{D}_0(\mathbf{d}^m)-\\ &\left[M_1^{q,n}(\mathbf{d}^m)\binom{\omega_q-1}{1}s^{\omega_q-2}+M_2^{q,n}(\mathbf{d}^m)\binom{\omega_q-2}{1}s^{\omega_q-3}+\ldots+M_{\omega_q-1}^{q,n}(\mathbf{d}^m)\right]\mathbf{D}_1(\mathbf{d}^m)+\\ &\left[M_1^{q,n}(\mathbf{d}^m)\binom{\omega_q-1}{2}s^{\omega_q-3}+M_2^{q,n}(\mathbf{d}^m)\binom{\omega_q-2}{2}s^{\omega_q-4}+\ldots+M_{\omega_q-2}^{q,n}(\mathbf{d}^m)\right]\mathbf{D}_2(\mathbf{d}^m)-\ldots+\\ &(-1)^{\omega_q}\left[M_1^{q,n}(\mathbf{d}^m)\binom{\omega_q-1}{\omega_q-2}s+M_2^{q,n}(\mathbf{d}^m)\right]\mathbf{D}_{\omega_q-2}(\mathbf{d}^m)-(-1)^{\omega_q}M_1^{q,n}(\mathbf{d}^m)\,\mathbf{D}_{\omega_q-1}(\mathbf{d}^m)=0,\end{aligned}\tag{5.22}$$

where we denote for short the following sums,

$$\mathbf{G}_0\left(\mathbf{d}^m\right) = 1 - \sum_{j=1}^{\beta_1(\mathbf{d}^m)} \cos\frac{2\pi n}{q} C_{j,1} + \ldots + (-1)^{m-1} \sum_{j=1}^{\beta_{m-1}(\mathbf{d}^m)} \cos\frac{2\pi n}{q} C_{j,m-1}\ , \tag{5.23}$$

$$\mathbf{G}_k\left(\mathbf{d}^m\right) = \sum_{j=1}^{\beta_1(\mathbf{d}^m)} C_{j,1}^k \cos\frac{2\pi n}{q} C_{j,1} - \ldots + (-1)^{m} \sum_{j=1}^{\beta_{m-1}(\mathbf{d}^m)} C_{j,m-1}^k \cos\frac{2\pi n}{q} C_{j,m-1}\ ,$$

$$\mathbf{D}_k\left(\mathbf{d}^m\right) = \sum_{j=1}^{\beta_1(\mathbf{d}^m)} C_{j,1}^k \sin\frac{2\pi n}{q} C_{j,1} - \ldots + (-1)^{m} \sum_{j=1}^{\beta_{m-1}(\mathbf{d}^m)} C_{j,m-1}^k \sin\frac{2\pi n}{q} C_{j,m-1}\ .$$

Equating the corresponding contributions coming from the power terms s^a, $a = 0, \ldots, \omega_q - 1$, in the l.h.s. and the r.h.s. of (5.22) we get a system of ω_q linear equations for $L_k^{q,n}\left(\mathbf{d}^m\right)$ and $M_k^{q,n}\left(\mathbf{d}^m\right)$. For short we present here the three first equations

$$a = \omega_q - 1 \qquad L_1^{q,n}\left(\mathbf{d}^m\right)\mathbf{G}_0\left(\mathbf{d}^m\right) + M_1^{q,n}\left(\mathbf{d}^m\right)\mathbf{D}_0\left(\mathbf{d}^m\right) = 0\ , \tag{5.24}$$

$$\begin{aligned} a = \omega_q - 2 \qquad & L_2^{q,n}\left(\mathbf{d}^m\right)\mathbf{G}_0\left(\mathbf{d}^m\right) + M_2^{q,n}\left(\mathbf{d}^m\right)\mathbf{D}_0\left(\mathbf{d}^m\right) + \\ & \binom{\omega_q - 1}{1}\left(L_1^{q,n}\left(\mathbf{d}^m\right)\mathbf{G}_1\left(\mathbf{d}^m\right) - M_1^{q,n}\left(\mathbf{d}^m\right)\mathbf{D}_1\left(\mathbf{d}^m\right)\right) = 0\ , \end{aligned} \tag{5.25}$$

$$\begin{aligned} a = \omega_q - 3 \qquad & L_3^{q,n}\left(\mathbf{d}^m\right)\mathbf{G}_0\left(\mathbf{d}^m\right) + M_3^{q,n}\left(\mathbf{d}^m\right)\mathbf{D}_0\left(\mathbf{d}^m\right) + \\ & \binom{\omega_q - 2}{1}\left(L_2^{q,n}\left(\mathbf{d}^m\right)\mathbf{G}_1\left(\mathbf{d}^m\right) - M_2^{q,n}\left(\mathbf{d}^m\right)\mathbf{D}_1\left(\mathbf{d}^m\right)\right) - \\ & \binom{\omega_q - 1}{2}\left(L_1^{q,n}\left(\mathbf{d}^m\right)\mathbf{G}_2\left(\mathbf{d}^m\right) - M_1^{q,n}\left(\mathbf{d}^m\right)\mathbf{D}_2\left(\mathbf{d}^m\right)\right) = 0\ , \end{aligned}$$

and the last one

$$\begin{aligned} a = 0 \qquad & L_{\omega_q}^{q,n}\left(\mathbf{d}^m\right)\mathbf{G}_0\left(\mathbf{d}^m\right) + M_{\omega_q}^{q,n}\left(\mathbf{d}^m\right)\mathbf{D}_0\left(\mathbf{d}^m\right) + \\ & L_{\omega_q-1}^{q,n}\left(\mathbf{d}^m\right)\mathbf{G}_1\left(\mathbf{d}^m\right) - M_{\omega_q-1}^{q,n}\left(\mathbf{d}^m\right)\mathbf{D}_1\left(\mathbf{d}^m\right) - \ldots - \\ & (-1)^{\omega_q}\left(L_2^{q,n}\left(\mathbf{d}^m\right)\mathbf{G}_{\omega_q-2}\left(\mathbf{d}^m\right) - M_2^{q,n}\left(\mathbf{d}^m\right)\mathbf{D}_{\omega_q-2}\left(\mathbf{d}^m\right)\right) + \\ & (-1)^{\omega_q}\left(L_1^{q,n}\left(\mathbf{d}^m\right)\mathbf{G}_{\omega_q-1}\left(\mathbf{d}^m\right) - M_1^{q,n}\left(\mathbf{d}^m\right)\mathbf{D}_{\omega_q-1}\left(\mathbf{d}^m\right)\right) = 0\ . \end{aligned}$$

The whole set of these equations can be written in more compact form

$$\begin{aligned} & L_k^{q,n}\left(\mathbf{d}^m\right)\mathbf{G}_0\left(\mathbf{d}^m\right) + M_k^{q,n}\left(\mathbf{d}^m\right)\mathbf{D}_0\left(\mathbf{d}^m\right) + \\ & \left(\omega_q - k + 1\right)\left(L_{k-1}^{q,n}\left(\mathbf{d}^m\right)\mathbf{G}_1\left(\mathbf{d}^m\right) - M_{k-1}^{q,n}\left(\mathbf{d}^m\right)\mathbf{D}_1\left(\mathbf{d}^m\right)\right) - \\ & \binom{\omega_q - k + 2}{2}\left(L_{k-2}^{q,n}\left(\mathbf{d}^m\right)\mathbf{G}_2\left(\mathbf{d}^m\right) - M_{k-2}^{q,n}\left(\mathbf{d}^m\right)\mathbf{D}_2\left(\mathbf{d}^m\right)\right) + \ldots - \\ & (-1)^k\binom{\omega_q - 2}{k-2}\left(L_2^{q,n}\left(\mathbf{d}^m\right)\mathbf{G}_{k-2}\left(\mathbf{d}^m\right) - M_2^{q,n}\left(\mathbf{d}^m\right)\mathbf{D}_{k-2}\left(\mathbf{d}^m\right)\right) + \\ & (-1)^k\binom{\omega_q - 1}{k-1}\left(L_1^{q,n}\left(\mathbf{d}^m\right)\mathbf{G}_{k-1}\left(\mathbf{d}^m\right) - M_1^{q,n}\left(\mathbf{d}^m\right)\mathbf{D}_{k-1}\left(\mathbf{d}^m\right)\right) = 0\ , \qquad 1 \le k \le \omega_q\ , \end{aligned} \tag{5.26}$$

where $L_k^{q,n}\left(\mathbf{d}^m\right) = M_k^{q,n}\left(\mathbf{d}^m\right) = 0$ if $k \le 0$.

5.3.2 $\mathbb{B}^{q,n}(s,\mathbf{d}^m)=0$

In this section we find the 2nd system of linear equations (5.31) for $L_k^{q,n}(\mathbf{d}^m)$ and $M_k^{q,n}(\mathbf{d}^m)$. Making use of Reps (5.19) and (5.20) substitute Reps (4.7) and (4.8) into expression for $\mathbb{B}^{q,n}(s,\mathbf{d}^m)$ given in section 5.3. Thus, we present the 2nd Eq. in (5.18) as follows,

$$\begin{aligned}&M_1^{q,n}(\mathbf{d}^m)\left[s^{\omega_q-1}-\mathbb{C}_{\omega_q-1}^{q,n}(s,\mathbf{d}^m)\right]+M_2^{q,n}(\mathbf{d}^m)\left[s^{\omega_q-2}-\mathbb{C}_{\omega_q-2}^{q,n}(s,\mathbf{d}^m)\right]+\ldots+\\&M_{\omega_q-1}^{q,n}(\mathbf{d}^m)\left[s-\mathbb{C}_1^{q,n}(s,\mathbf{d}^m)\right]+M_{\omega_q}^{q,n}(\mathbf{d}^m)\left[1-\mathbb{C}_0^{q,n}(s,\mathbf{d}^m)\right]-\\&L_1^{q,n}(\mathbf{d}^m)\,\mathbb{S}_{\omega_q-1}^{q,n}(s,\mathbf{d}^m)-L_2^{q,n}(\mathbf{d}^m)\,\mathbb{S}_{\omega_q-2}^{q,n}(s,\mathbf{d}^m)-\ldots-L_{\omega_q}^{q,n}(\mathbf{d}^m)\,\mathbb{S}_0^{q,n}(s,\mathbf{d}^m)=0\,.\end{aligned}\tag{5.27}$$

Keeping in mind (5.23), after binomial expansion of the terms $(s-C_{j,i})^p$ in (5.19) and (5.20) and subsequent substitution into (5.27) we recast the power terms in s and get the final equality

$$\begin{aligned}&\left[M_1^{q,n}(\mathbf{d}^m)\,s^{\omega_q-1}+M_2^{q,n}(\mathbf{d}^m)\,s^{\omega_q-2}+\ldots+M_{\omega_q}^{q,n}(\mathbf{d}^m)\right]\mathbf{G}_0(\mathbf{d}^m)+\\&\left[M_1^{q,n}(\mathbf{d}^m)\binom{\omega_q-1}{1}s^{\omega_q-2}+M_2^{q,n}(\mathbf{d}^m)\binom{\omega_q-2}{1}s^{\omega_q-3}+\ldots+M_{\omega_q-1}^{q,n}(\mathbf{d}^m)\right]\mathbf{G}_1(\mathbf{d}^m)-\\&\left[M_1^{q,n}(\mathbf{d}^m)\binom{\omega_q-1}{2}s^{\omega_q-3}+M_2^{q,n}(\mathbf{d}^m)\binom{\omega_q-2}{2}s^{\omega_q-4}+\ldots+M_{\omega_q-2}^{q,n}(\mathbf{d}^m)\right]\mathbf{G}_2(\mathbf{d}^m)+\ldots-\\&(-1)^{\omega_q}\left[M_1^{q,n}(\mathbf{d}^m)\binom{\omega_q-1}{\omega_q-2}s+M_2^{q,n}(\mathbf{d}^m)\right]\mathbf{G}_{\omega_q-2}(\mathbf{d}^m)+(-1)^{\omega_q}\,M_1^{q,n}(\mathbf{d}^m)\,\mathbf{G}_{\omega_q-1}(\mathbf{d}^m)-\\&\left[L_1^{q,n}(\mathbf{d}^m)\,s^{\omega_q-1}+L_2^{q,n}(\mathbf{d}^m)\,s^{\omega_q-2}+\ldots+L_{\omega_q}^{q,n}(\mathbf{d}^m)\right]\mathbf{D}_0(\mathbf{d}^m)+\\&\left[L_1^{q,n}(\mathbf{d}^m)\binom{\omega_q-1}{1}s^{\omega_q-2}+L_2^{q,n}(\mathbf{d}^m)\binom{\omega_q-2}{1}s^{\omega_q-3}+\ldots+L_{\omega_q-1}^{q,n}(\mathbf{d}^m)\right]\mathbf{D}_1(\mathbf{d}^m)-\\&\left[L_1^{q,n}(\mathbf{d}^m)\binom{\omega_q-1}{2}s^{\omega_q-3}+L_2^{q,n}(\mathbf{d}^m)\binom{\omega_q-2}{2}s^{\omega_q-4}+\ldots+L_{\omega_q-2}^{q,n}(\mathbf{d}^m)\right]\mathbf{D}_2(\mathbf{d}^m)+\ldots-\\&(-1)^{\omega_q}\left[L_1^{q,n}(\mathbf{d}^m)\binom{\omega_q-1}{\omega_q-2}s+L_2^{q,n}(\mathbf{d}^m)\right]\mathbf{D}_{\omega_q-2}(\mathbf{d}^m)+(-1)^{\omega_q}L_1^{q,n}(\mathbf{d}^m)\,\mathbf{D}_{\omega_q-1}(\mathbf{d}^m)=0.\end{aligned}\tag{5.28}$$

Equating the corresponding contributions coming from the power terms s^a, $a=0,\ldots,\omega_q-1$, in the l.h.s. and the r.h.s. of (5.28) we get the 2nd system of ω_q linear equations for $L_k^{q,n}(\mathbf{d}^m)$ and $M_k^{q,n}(\mathbf{d}^m)$. For short we present here the three first equations

$$a=\omega_q-1\qquad M_1^{q,n}(\mathbf{d}^m)\,\mathbf{G}_0(\mathbf{d}^m)-L_1^{q,n}(\mathbf{d}^m)\,\mathbf{D}_0(\mathbf{d}^m)=0\,,\tag{5.29}$$

$$a=\omega_q-2\qquad \begin{aligned}&M_2^{q,n}(\mathbf{d}^m)\,\mathbf{G}_0(\mathbf{d}^m)-L_2^{q,n}(\mathbf{d}^m)\,\mathbf{D}_0(\mathbf{d}^m)+\\&\binom{\omega_q-1}{1}\left(M_1^{q,n}(\mathbf{d}^m)\,\mathbf{G}_1(\mathbf{d}^m)+L_1^{q,n}(\mathbf{d}^m)\,\mathbf{D}_1(\mathbf{d}^m)\right)=0\,,\end{aligned}\tag{5.30}$$

$$a=\omega_q-3\qquad \begin{aligned}&M_3^{q,n}(\mathbf{d}^m)\,\mathbf{G}_0(\mathbf{d}^m)-L_3^{q,n}(\mathbf{d}^m)\,\mathbf{D}_0(\mathbf{d}^m)+\\&\binom{\omega_q-2}{1}\left(M_2^{q,n}(\mathbf{d}^m)\,\mathbf{G}_1(\mathbf{d}^m)+L_2^{q,n}(\mathbf{d}^m)\,\mathbf{D}_1(\mathbf{d}^m)\right)-\\&\binom{\omega_q-1}{2}\left(M_1^{q,n}(\mathbf{d}^m)\,\mathbf{G}_2(\mathbf{d}^m)+L_1^{q,n}(\mathbf{d}^m)\,\mathbf{D}_2(\mathbf{d}^m)\right)=0\,,\end{aligned}$$

and the last one

$$
\begin{aligned}
a=0 \qquad & M_{\omega_q}^{q,n}\left(\mathbf{d}^m\right)\mathbf{G}_0\left(\mathbf{d}^m\right)-L_{\omega_q}^{q,n}\left(\mathbf{d}^m\right)\mathbf{D}_0\left(\mathbf{d}^m\right)+\\
& M_{\omega_q-1}^{q,n}\left(\mathbf{d}^m\right)\mathbf{G}_1\left(\mathbf{d}^m\right)+L_{\omega_q-1}^{q,n}\left(\mathbf{d}^m\right)\mathbf{D}_1\left(\mathbf{d}^m\right)-\ldots+\\
& (-1)^{\omega_q}\left(M_2^{q,n}\left(\mathbf{d}^m\right)\mathbf{G}_{\omega_q-2}\left(\mathbf{d}^m\right)+L_2^{q,n}\left(\mathbf{d}^m\right)\mathbf{D}_{\omega_q-2}\left(\mathbf{d}^m\right)\right)+\\
& (-1)^{\omega_q}\left(M_1^{q,n}\left(\mathbf{d}^m\right)\mathbf{G}_{\omega_q-1}\left(\mathbf{d}^m\right)+L_1^{q,n}\left(\mathbf{d}^m\right)\mathbf{D}_{\omega_q-1}\left(\mathbf{d}^m\right)\right)=0\ .
\end{aligned}
$$

The whole set of these equations can be written in more compact form

$$
\begin{aligned}
& M_k^{q,n}\left(\mathbf{d}^m\right)\mathbf{G}_0\left(\mathbf{d}^m\right)-L_k^{q,n}\left(\mathbf{d}^m\right)\mathbf{D}_0\left(\mathbf{d}^m\right)+\\
& (\omega_q-k+1)\left(M_{k-1}^{q,n}\left(\mathbf{d}^m\right)\mathbf{G}_1\left(\mathbf{d}^m\right)+L_{k-1}^{q,n}\left(\mathbf{d}^m\right)\mathbf{D}_1\left(\mathbf{d}^m\right)\right)-\\
& \binom{\omega_q-k+2}{2}\left(M_{k-2}^{q,n}\left(\mathbf{d}^m\right)\mathbf{G}_2\left(\mathbf{d}^m\right)+L_{k-2}^{q,n}\left(\mathbf{d}^m\right)\mathbf{D}_2\left(\mathbf{d}^m\right)\right)+\ldots-\\
& (-1)^k\binom{\omega_q-2}{k-2}\left(M_2^{q,n}\left(\mathbf{d}^m\right)\mathbf{G}_{k-2}\left(\mathbf{d}^m\right)+L_2^{q,n}\left(\mathbf{d}^m\right)\mathbf{D}_{k-2}\left(\mathbf{d}^m\right)\right)+\\
& (-1)^k\binom{\omega_q-1}{k-1}\left(M_1^{q,n}\left(\mathbf{d}^m\right)\mathbf{G}_{k-1}\left(\mathbf{d}^m\right)+L_1^{q,n}\left(\mathbf{d}^m\right)\mathbf{D}_{k-1}\left(\mathbf{d}^m\right)\right)=0\ , \qquad 1\le k\le\omega_q\ ,
\end{aligned}
\tag{5.31}
$$

where $L_k^{q,n}\left(\mathbf{d}^m\right)=M_k^{q,n}\left(\mathbf{d}^m\right)=0$ if $k\le 0$.

5.3.3 Common Solutions of Equations $\mathbb{A}^{q,n}\left(s,\mathbf{d}^m\right)=0$ and $\mathbb{B}^{q,n}\left(s,\mathbf{d}^m\right)=0$

In this section we finish a proof of Theorem 2 stated in section 2.1. For this purpose we combine two sets of linear equations (5.26) and (5.31) for $\mathbf{G}_r\left(\mathbf{d}^m\right)$ and $\mathbf{D}_r\left(\mathbf{d}^m\right)$ and solve them together. Start with two linear equations (5.24) and (5.29) for $\mathbf{G}_0\left(\mathbf{d}^m\right)$ and $\mathbf{D}_0\left(\mathbf{d}^m\right)$,

$$
L_1^{q,n}\left(\mathbf{d}^m\right)\mathbf{G}_0\left(\mathbf{d}^m\right)+M_1^{q,n}\left(\mathbf{d}^m\right)\mathbf{D}_0\left(\mathbf{d}^m\right)=0, \qquad M_1^{q,n}\left(\mathbf{d}^m\right)\mathbf{G}_0\left(\mathbf{d}^m\right)-L_1^{q,n}\left(\mathbf{d}^m\right)\mathbf{D}_0\left(\mathbf{d}^m\right)=0,
$$

with discriminant $\Delta_1^{q,n}\left(\mathbf{d}^m\right)=\left[L_1^{q,n}\left(\mathbf{d}^m\right)\right]^2+\left[M_1^{q,n}\left(\mathbf{d}^m\right)\right]^2$. Noting that by (4.21), or by Lemma 1, both $L_1^{q,n}\left(\mathbf{d}^m\right)$ and $M_1^{q,n}\left(\mathbf{d}^m\right)$ do not vanish simultaneously, we arrive at trivial solution

$$
\mathbf{G}_0\left(\mathbf{d}^m\right)=\mathbf{D}_0\left(\mathbf{d}^m\right)=0\ . \tag{5.32}
$$

Keeping in mind (5.32) continue with two equations (5.25) and (5.30) for $\mathbf{G}_1\left(\mathbf{d}^m\right)$ and $\mathbf{D}_1\left(\mathbf{d}^m\right)$,

$$
L_1^{q,n}\left(\mathbf{d}^m\right)\mathbf{G}_1\left(\mathbf{d}^m\right)+M_1^{q,n}\left(\mathbf{d}^m\right)\mathbf{D}_1\left(\mathbf{d}^m\right)=0, \qquad M_1^{q,n}\left(\mathbf{d}^m\right)\mathbf{G}_1\left(\mathbf{d}^m\right)-L_1^{q,n}\left(\mathbf{d}^m\right)\mathbf{D}_1\left(\mathbf{d}^m\right)=0,
$$

with the same discriminant $\Delta_1^{q,n}\left(\mathbf{d}^m\right)\neq 0$ discussed above. By the same reason as for (5.32) we obtain $\mathbf{G}_1\left(\mathbf{d}^m\right)=\mathbf{D}_1\left(\mathbf{d}^m\right)=0$. Prove by induction that

$$
\mathbf{G}_r\left(\mathbf{d}^m\right)=\mathbf{D}_r\left(\mathbf{d}^m\right)=0\ , \quad 1\le r<\omega_q\ . \tag{5.33}
$$

Indeed, let $\mathbf{G}_r\left(\mathbf{d}^m\right)=\mathbf{D}_r\left(\mathbf{d}^m\right)=0$, $1\le r<r_*$ and write two linear equations (5.26) and (5.31) for $k=r_*+1$. Simplifying both equations we get

$$
L_1^{q,n}\left(\mathbf{d}^m\right)\mathbf{G}_{r_*}\left(\mathbf{d}^m\right)+M_1^{q,n}\left(\mathbf{d}^m\right)\mathbf{D}_{r_*}\left(\mathbf{d}^m\right)=0, \quad M_1^{q,n}\left(\mathbf{d}^m\right)\mathbf{G}_{r_*}\left(\mathbf{d}^m\right)-L_1^{q,n}\left(\mathbf{d}^m\right)\mathbf{D}_{r_*}\left(\mathbf{d}^m\right)=0,
$$

with discriminant $\Delta_1^{q,n}(\mathbf{d}^m) \neq 0$. Thus, we arrive at trivial solution $\mathbf{G}_{r_*}(\mathbf{d}^m) = \mathbf{D}_{r_*}(\mathbf{d}^m) = 0$, and therefore the existence of the general solutions (5.33) is proven.

On basis of $\mathbf{G}_r(\mathbf{d}^m)$ and $\mathbf{D}_r(\mathbf{d}^m)$ build two complex expressions and make use of (5.33),

$$\mathbf{G}_0(\mathbf{d}^m) - i\,\mathbf{D}_0(\mathbf{d}^m) = 0\,, \qquad \mathbf{G}_r(\mathbf{d}^m) + i\,\mathbf{D}_r(\mathbf{d}^m) = 0\,, \quad 1 \le r < \omega_q\,. \tag{5.34}$$

By definition (5.23) write two equalities (5.34) in exponential form and arrive at two equalities of Theorem 2. Thus, this finishes proof of Theorem 2.

We finish this section calculating the total number $N(\mathbf{d}^m)$ of polynomial and quasipolynomial identities imposed on the syzygies degrees of numerical semigroup $\mathsf{S}(\mathbf{d}^m)$. By Theorem 1 we have m polynomial identities including (2.5). By Theorem 2 for any pair n and $q \ge 2$ we can write ω_q quasipolynomial identities such that $\omega_q = 0$ if $q \nmid d_j$, $1 \le j \le m$. Keeping in mind a range of valuation for n, $\gcd(n,q) = 1$ and $1 \le n < q/2$, we have for a fixed $q \ge 3$ exactly $\phi(q)/2$ values of n, where $\phi(q)$ stands for the totient function. The value $q = 2$ is a special one and by (5.16) it corresponds to the value $n = 1$ only. Since q has to divide at least one generator d_j of the tuple $\mathbf{d}^m = \{d_1, \ldots, d_m\}$ we arrive finally at

$$N(\mathbf{d}^m) = m + \omega_2 + \frac{1}{2}\sum_{q=3}^{\max \mathbf{d}^m} \omega_q \phi(q)\,. \tag{5.35}$$

6 Applications

In this section we discuss different applications of Theorems 1 and 2 to the various kind of numerical semigroups and find more compact form of polynomial identities. We illustrate a validity of identities by examples for numerical semigroups discussed earlier in literature.

6.1 Complete Intersections

Start with a simple case of numerical semigroups where its Hilbert series is given by (2.11) and a power sums of the degrees of syzygies read for $1 \le k < m$,

$$\sum_{j=1}^{\beta_1} C_{j,1}^k = \sum_{j=1}^{m-1} e_j^k\,, \quad \sum_{j=1}^{\beta_2} C_{j,2}^k = \sum_{j>k=1}^{m-1} (e_j + e_r)^k\,, \quad \ldots, \quad C_{1,m-1}^k = E^k\,, \quad E = \sum_{j=1}^{m-1} e_j\,. \tag{6.1}$$

Substituting (6.1) into (2.12) in Theorem 1 we get a set of algebraic identities for the tuple of integers $(e_1, \ldots, e_{m-1})$ when $1 \le k \le m-2$,

$$\sum_{j=1}^{m-1} e_j^k - \sum_{j>k=1}^{m-1} (e_j + e_r)^k + \sum_{j>k>l=1}^{m-1} (e_j + e_r + e_l)^k - \ldots + (-1)^m \left(\sum_{j=1}^{m-1} e_j\right)^k = 0\,. \tag{6.2}$$

Identity (6.2) looks very *'combinatorial'* and, indeed, for $k = 1$ it can be verified trivially,

$$\sum_{j=1}^{m-1} e_j - \sum_{j>k=1}^{m-1} (e_j + e_r) + \ldots + (-1)^m E = \left[1 - \binom{m-2}{1} + \ldots + (-1)^m\right] E = (1-1)^{m-2} E = 0\,.$$

However, for $1<k<m-1$ such straightforward way of verification is hardly performable. This is why in Appendix A we give its verification based on the inclusion-exclusion principle applied to the set of tuples of length k comprised elements of $m-1$ different sorts [1].

What is much more important that this approach allows to calculate the left hand side in (6.2) when $k=m-1$. Namely, the following identity holds,

$$\sum_{j=1}^{m-1} e_j^{m-1} - \sum_{j>k=1}^{m-1} (e_j+e_r)^{m-1} + \ldots + (-1)^m \left(\sum_{j=1}^{m-1} e_j\right)^{m-1} = (-1)^m (m-1)! \prod_{i=1}^{m-1} e_i \,. \quad (6.3)$$

Combining now the last identity with the 2nd identity (2.13) in Theorem 1 we get finally

Corollary 2 *Let the complete intersection semigroup $\mathsf{S}\left(\mathbf{d}^m\right)$ be given with its Hilbert series $H\left(\mathbf{d}^m;z\right)$ in accordance with (2.2) and (2.11). Then the following identity holds,*

$$\prod_{i=1}^{m-1} e_i = \prod_{i=1}^{m} d_i \,. \quad (6.4)$$

In other words, the entire set of $m-1$ polynomial identities of Theorem 1 is reduced to only one nontrivial identity (6.4). Regarding Theorem 2, its quasipolynomial relations imply another set of identities which definitely do not have combinatorial origin.

Corollary 3 *Let the complete intersection semigroup $\mathsf{S}\left(\mathbf{d}^m\right)$ be given with its Hilbert series $H\left(\mathbf{d}^m;z\right)$ in accordance with (2.2) and (2.11). Then for every $1<q\leq \max\{d_1,\ldots,d_m\}$, and $\gcd(n,q)=1$, $1\leq n<q/2$, the following quasipolynomial identities hold,*

$$\sum_{j=1}^{m-1} e_j^r \exp\left(i\frac{2\pi n}{q}e_j\right) - \sum_{j>l=1}^{m-1}(e_j+e_l)^r \exp\left(i\frac{2\pi n}{q}(e_j+e_l)\right) + \ldots + (-1)^m \sum_{j=1}^{m-1}(E-e_j)^r \exp\left(i\frac{2\pi n}{q}(E-e_j)\right) = 0.$$

where $1\leq r<\omega_q$ and $E=\sum_{i=1}^{m-1}e_i$. However, in the case $r=0$ another identity holds,

$$\sum_{j=1}^{m-1} \exp\left(i\frac{2\pi n}{q}e_j\right) - \sum_{j>l=1}^{m-1} \exp\left(i\frac{2\pi n}{q}(e_j+e_l)\right) + \ldots + (-1)^m \sum_{j=1}^{m-1} \exp\left(i\frac{2\pi n}{q}(E-e_j)\right) = 1\,.$$

Below we illustrate Corollaries 2 and 3 by presenting the Hilbert series of two complete intersection semigroups generated by four elements [11], [23]: a) $(8,9,10,12)$ and b) $(10,14,15,21)$,

$$H_a(z) = \frac{(1-z^{18})(1-z^{20})(1-z^{24})}{(1-z^8)(1-z^9)(1-z^{10})(1-z^{12})}, \quad H_b(z) = \frac{(1-z^{30})(1-z^{35})(1-z^{42})}{(1-z^{10})(1-z^{14})(1-z^{15})(1-z^{21})} \,.$$

For both semigroups the identities of both Corollaries 2 and 3 are satisfied.

6.1.1 Telescopic Semigroups

Two semigroups $\mathsf{S}\left(\{8,9,10,12\}\right)$ and $\mathsf{S}\left(\{10,14,15,21\}\right)$ discussed in the previous section 6.1 present two different kinds of complete intersections: telescopic and complete intersections-not

[1] Combinatorial proof of (6.2) was kindly communicated to me by R. Pinchasi (Technion).

telescopic semigroups, respectively. The semigroups of the 1st kind present the most simple sort of complete intersections.

Following [21] start with definition. For given numerical semigroup $\mathsf{S}\left(\mathbf{d}^m\right)$ with generators $\mathbf{d}^m = \{d_1,\ldots,d_m\}$, $\gcd(d_1,\ldots,d_m) = 1$, (not necessarily in increasing order), let us denote $g_k = \gcd(d_1,\ldots,d_k) > 1$ and $\mathsf{S}_k = \mathsf{S}\left(\left\{\frac{d_1}{g_k},\frac{d_2}{g_k},\ldots,\frac{d_k}{g_k}\right\}\right)$ for $1 \le k < m$, and $g_1 = d_1$, $g_m = 1$. Then $\mathsf{S}\left(\mathbf{d}^m\right)$ is said to be telescopic iff $\frac{d_k}{g_k} \in \mathsf{S}_{k-1}$ for all $2 \le k \le m$. Regarding its Hilbert series (2.11), the syzygy degrees e_j of the 1st kind follow by calculating $m-1$ minimal linear relations between the generators d_j. Their straightforward calculation gives,

$$e_1 = lcm(d_1,d_2)\ , \quad e_2 = d_3\frac{g_2}{g_3}\ ,\ldots,\quad e_j = d_{j+1}\frac{g_j}{g_{j+1}}\ ,\quad 2 \le j < m\ ,\quad e_{m-1} = d_m\, g_{m-1}\ . \tag{6.5}$$

As for the polynomial identity (6.4), it becomes trivial in the case of telescopic semigroups. Indeed, this can be verified if substituting (6.5) into (6.4). Thus, there remain only the set of quasipolynomial identities after substitution (6.4) into identities of Corollary 3.

6.2 Symmetric Numerical Semigroups

This kind of semigroups is of high importance due to the theorem of Kunz [24] which asserts that a graded polynomial subring, associated with semigroup $\mathsf{S}\left(\mathbf{d}^m\right)$, is Gorenstein iff $\mathsf{S}\left(\mathbf{d}^m\right)$ is symmetric. For short, denote $\mathcal{Q}\left(\mathbf{d}^m\right) = \deg Q\left(\mathbf{d}^m;z\right)$ and define the following combination of Betti numbers,

$$\mathcal{B}\left(\mathbf{d}^m\right) = -1 + \beta_1\left(\mathbf{d}^m\right) - \beta_2\left(\mathbf{d}^m\right) + \ldots + (-1)^{\mu}\beta_{\mu-1}\left(\mathbf{d}^m\right)\ ,\quad \mu = \left\lfloor\frac{m}{2}\right\rfloor\ . \tag{6.6}$$

Then, by Theorem 1 for symmetric semigroups with even edim $2m$ it holds for $1 \le k \le 2m-1$,

$$\mathcal{Q}^k\left(\mathbf{d}^{2m}\right) - \sum_{r=1}^{m-1}(-1)^r\sum_{j=1}^{\beta_r\left(\mathbf{d}^{2m}\right)}\left\{C_{j,r}^k - \left[\mathcal{Q}\left(\mathbf{d}^{2m}\right) - C_{j,r}\right]^k\right\} = (2m-1)!\ \pi_{2m}\ \delta_{k,2m-1}\ . \tag{6.7}$$

In the case $k=1$ the last identity reads gives

$$\sum_{j=1}^{\beta_1\left(\mathbf{d}^{2m}\right)} C_{j,1} - \sum_{j=1}^{\beta_2\left(\mathbf{d}^{2m}\right)} C_{j,2} + \ldots + (-1)^m \sum_{j=1}^{\beta_{m-1}\left(\mathbf{d}^{2m}\right)} C_{j,m-1} = \frac{1}{2}\mathcal{B}\left(\mathbf{d}^{2m}\right)\mathcal{Q}\left(\mathbf{d}^{2m}\right)\ . \tag{6.8}$$

By the same Theorem 1 for symmetric semigroups with odd edim=$2m+1$ it holds for $1 \le k \le 2m$,

$$\mathcal{Q}^k\left(\mathbf{d}^{2m+1}\right) + \sum_{r=1}^{m-1}(-1)^r\sum_{j=1}^{\beta_r\left(\mathbf{d}^{2m+1}\right)}\left\{C_{j,r}^k + \left[\mathcal{Q}\left(\mathbf{d}^{2m+1}\right) - C_{j,r}\right]^k\right\} + (-1)^m\sum_{j=1}^{\beta_m\left(\mathbf{d}^{2m+1}\right)} C_{j,m}^k +$$
$$(2m)!\ \pi_{2m+1}\ \delta_{k,2m} = 0\ , \tag{6.9}$$

that in the case $k=1$ gives

$$\sum_{j=1}^{\beta_m\left(\mathbf{d}^{2m+1}\right)} C_{j,m} = \mathcal{B}\left(\mathbf{d}^{2m+1}\right)\mathcal{Q}\left(\mathbf{d}^{2m+1}\right). \tag{6.10}$$

6.2.1 Symmetric Semigroups $\mathsf{S}\left(\mathbf{d}^4\right)$ (not complete intersections)

In 1975, Bresinsky [6] has shown that symmetric semigroups $\mathsf{S}\left(\mathbf{d}^4\right)$, which are not complete intersections, have always $\beta_1\left(\mathbf{d}^4\right)=5$. Denoting invariants $I_k=\sum_{j=1}^{5} C_{j,1}^k$, we obtain by (6.7) two different polynomial identities,

$$8I_3-6I_2I_1+I_1^3=24\pi_4\ , \quad \mathcal{Q}\left(\mathbf{d}^4\right)=\frac{1}{2}\,I_1\ . \tag{6.11}$$

Below we illustrate identities (6.11) by presenting the Hilbert series of two symmetric semigroups (not complete intersection) generated by four elements [11],

$$H\left(\{5,6,7,8\};z\right)=\frac{1-z^{12}-z^{13}-z^{14}-z^{15}-z^{16}+z^{19}+z^{20}+z^{21}+z^{22}+z^{23}-z^{35}}{\left(1-z^5\right)\left(1-z^6\right)\left(1-z^7\right)\left(1-z^8\right)}\ ,$$

$$H\left(\{8,13,15,17\};z\right)=\frac{1-z^{30}-z^{32}-z^{34}-z^{39}-z^{41}+z^{47}+z^{49}+z^{54}+z^{56}+z^{58}-z^{88}}{\left(1-z^8\right)\left(1-z^{13}\right)\left(1-z^{15}\right)\left(1-z^{17}\right)}\ .$$

For both semigroups the quasipolynomial identities of Theorem 2 are also satisfied.

6.2.2 Symmetric Semigroups $\mathsf{S}\left(\mathbf{d}^5\right)$ (not complete intersections)

In contrast to symmetric semigroups $\mathsf{S}\left(\mathbf{d}^4\right)$, the problem of admissible values of the 1st Betti number for symmetric semigroups (not complete intersections) $\mathsf{S}\left(\mathbf{d}^m\right)$, $m\geq 5$, or their upper bounds, are still open (see [5], Problem 3). Therefore in this section we study the polynomial identities of Theorem 1 for $m=5$ and arbitrary $\beta_1\left(\mathbf{d}^5\right)$.

Henceforth, we skip for short the notation $\mathbf{d}^5$ in the Betti numbers and denote symmetric invariants $J_{r,k}=\sum_{j=1}^{\beta_r} C_{j,r}^k$ where $r=1,2$ and $1\leq k\leq 4$. By (6.9) and (6.10) we get four different polynomial identities,

$$J_{2,1}=(\beta_1-1)\mathcal{Q}\left(\mathbf{d}^5\right), \qquad J_{2,1}(2J_{1,1}-J_{2,1})+(\beta_1-1)\left(J_{2,2}-2J_{1,2}\right)=0, \tag{6.12}$$

$$J_{2,1}^2(3J_{1,1}-J_{2,1})-3(\beta_1-1)J_{1,2}J_{2,1}+(\beta_1-1)^2J_{2,3}=0,$$

$$J_{2,1}^3(4J_{1,1}-J_{2,1})-6(\beta_1-1)J_{1,2}J_{2,1}^2+4(\beta_1-1)^2J_{1,3}J_{2,1}+(\beta_1-1)^3\left(J_{2,4}-2J_{1,4}-24\pi_5\right)=0.$$

For verification we have chosen the known symmetric semigroup $\mathsf{S}\left(\{19,23,29,31,37\}\right)$ generated by five elements and found by computer calculations by Bresinsky [7]. Its corresponding Betti numbers read: $\beta_1=13,\ \beta_2=24,\ \beta_3=13,\ \beta_4=1$. The following numerator of its Hilbert series was calculated by the author,

$$\begin{aligned} Q(z) \;=\;& 1-z^{60}-z^{69}-z^{75}-z^{77}-z^{81}-z^{85}-z^{87}-z^{93}-z^{95}+z^{98}-z^{99}+z^{100}-z^{103}+\\ & z^{104}-z^{105}+2z^{106}+z^{108}+z^{110}-z^{111}+z^{112}+z^{114}+z^{116}+2z^{118}+2z^{122}+\\ & z^{124}+z^{126}+z^{128}-z^{129}+z^{130}+z^{132}+2z^{134}-z^{135}+z^{136}-z^{137}+z^{140}-z^{141}+\\ & z^{142}-z^{145}-z^{147}-z^{153}-z^{155}-z^{159}-z^{163}-z^{165}-z^{171}-z^{180}+z^{240}\ . \end{aligned}$$

Straightforward calculation shows that the polynomial identities (6.12) are satisfied as well as the quasipolynomial identities of Theorem 2.

6.3 Nonsymmetric Numerical Semigroups $\mathsf{S}\left(\mathbf{d}^3\right)$

In 2004, studying the Frobenius problem for numerical semigroups $\mathsf{S}\left(\mathbf{d}^3\right)$ with the Hilbert series

$$H\left(\mathbf{d}^3;z\right)=\frac{1-z^{e_1}-z^{e_2}-z^{e_3}+z^{q_1}+z^{q_2}}{\left(1-z^{d_1}\right)\left(1-z^{d_2}\right)\left(1-z^{d_3}\right)}\;, \tag{6.13}$$

in [27] and [14] there were found the degrees q_1 and q_2, $q_1<q_2$, of syzygies of the 2nd kind in terms of 3 degrees e_1, e_2 and e_3 of syzygies of the 1st kind and 3 generators d_1,d_2,d_3,

$$q_{1,2}=\frac{1}{2}\left[(e_1+e_2+e_3)\pm\sqrt{e_1^2+e_2^2+e_3^2-2(e_1e_2+e_2e_3+e_3e_1)+4d_1d_2d_3}\;\right]\;. \tag{6.14}$$

The last formula gives by (2.7) the Frobenius number $F\left(\mathbf{d}^3\right)=q_2-\sigma_1$. Recently a derivation of formula (6.14) has been shorten essentially [3] by making use of the Apéry set of $\mathsf{S}\left(\mathbf{d}^3\right)$ and of relation between its generating function and the Hilbert series. It turns out that the way to derive (6.14) can be shorten much more. Indeed, by Theorem 1 two identities hold

$$e_1+e_2+e_3=q_1+q_2\;,\qquad e_1^2+e_2^2+e_3^2=q_1^2+q_2^2-2d_1d_2d_3\;, \tag{6.15}$$

and after trivial algebra we arrive at (6.14).

The derivation of the degrees e_1, e_2 and e_3 through the three generators d_i is much more difficult problem which encounters the Curtis theorem [9] on non algebraic Rep of the Frobenius number $F\left(\mathbf{d}^3\right)$. By (6.14) this statement is equivalent to the claim that neither of the syzygy degrees e_i is representable by algebraic function in d_1,d_2,d_3. The analytic Rep for e_i by integration in the complex plane was found in [15]. However, based on quasipolynomial identities of Theorem 2 one can build another set of non algebraic Reps. E.g., in the case when all generators d_i are primes the syzygy degrees e_i come as solutions of polynomial equations (6.15) together with exponential equations

$$\xi_{d_j}^{ne_1}+\xi_{d_j}^{ne_2}+\xi_{d_j}^{ne_3}=\xi_{d_j}^{nq_1}+\xi_{d_j}^{nq_2}+1\;,\quad j=1,2,3\;,\quad \gcd(n,d_j)=1\;. \tag{6.16}$$

6.3.1 Pseudosymmetric Semigroups $\mathsf{S}\left(\mathbf{d}^3\right)$

A numerical semigroup $\mathsf{S}\left(\mathbf{d}^m\right)$ is pseudosymmetric if $F\left(\mathbf{d}^m\right)$ is even and the only integer such $s\in\mathbb{N}\setminus\mathsf{S}\left(\mathbf{d}^m\right)$ and $F\left(\mathbf{d}^m\right)-s\notin\mathsf{S}\left(\mathbf{d}^m\right)$ is $s=1/2\;F\left(\mathbf{d}^m\right)$. A case $m=3$ is most simple and allows to calculate the Frobenius number [28], [17],

$$F\left(\mathbf{d}^3\right)=-\sigma_1+\sqrt{\sigma_1^2-4(d_1d_2+d_2d_3+d_3d_1)+4d_1d_2d_3}\;, \tag{6.17}$$

and the whole numerator of the Hilbert series [17],

$$Q\left(\mathbf{d}^3;z\right)=1-z^{d_1+d_2+\frac{1}{2}F\left(\mathbf{d}^3\right)}-z^{d_2+d_3+\frac{1}{2}F\left(\mathbf{d}^3\right)}-z^{d_3+d_1+\frac{1}{2}F\left(\mathbf{d}^3\right)}+z^{\frac{1}{2}F\left(\mathbf{d}^3\right)+\sigma_1}+z^{F\left(\mathbf{d}^3\right)+\sigma_1}\;.$$

The degrees of syzygies of the above expression satisfy the 1st Eq. in (6.15). As for the 2nd Eq. in (6.15), it is reduced to quadratic Eq. in $F\left(\mathbf{d}^3\right)$ and gives (6.17).

6.4 Numerical Semigroups of Maximal edim

A semigroup of maximal edim (for short, MED semigroup), which is generated by tuple $\mathbf{d}^m_{MED} = \{m, d_2, \ldots, d_m\}$, is never symmetric. Many explicit results are known about its Betti numbers, genus and Frobenius number [20], [30], [2], e.g., $F(\mathbf{d}^m_{MED}) = d_m - m$. Regarding the Hilbert series, which was found recently [17], the partial contributions $Q_k(\mathbf{d}^m_{MED}; z)$ (see (2.3)) to the whole numerator $Q(\mathbf{d}^m_{MED}; z)$ in Hilbert series (2.2) read,

$$Q_k(\mathbf{d}^m_{MED}; z) = I_{m,k}(z) + J_{m,k}(z)\ , \quad 1 \le k \le m-2\ , \quad Q_{m-1}(\mathbf{d}^m_{MED}; z) = I_{m,m-1}(z)\ , \quad (6.18)$$

where for $d_j \in \{d_2, \ldots, d_m\}$ the following notations stand,

$$I_{m,k}(z) = \sum_{j_1 \neq j_2 > \ldots > j_{k-1} \ge 2}^{m} z^{2d_{j_1} + \overbrace{d_{j_2} + \ldots + d_{j_k}}^{\text{k}-1\ terms}}\ , \quad J_{m,k}(z) = k \sum_{j_1 > \ldots > j_k \ge 2}^{m} z^{\overbrace{d_{j_1} + d_{j_2} + \ldots + d_{j_{k+1}}}^{\text{k}+1\ terms}}. \quad (6.19)$$

We give the polynomials $I_{m,k}(z)$ and $J_{m,k}(z)$ for small and large indices k,

$$I_{m,1}(z) = \sum_{j_1 \ge 2}^{m} z^{2d_{j_1}}, \quad I_{m,2}(z) = \sum_{j_1 \neq j_2 \ge 2}^{m} z^{2d_{j_1} + d_{j_2}}, \quad I_{m,3}(z) = \sum_{j_1 \neq j_2 > j_3 \ge 2}^{m} z^{2d_{j_1} + d_{j_2} + d_{j_3}}, \ \ldots,$$

$$I_{m,m-2}(z) = \sum_{j_1 \neq j_2 > \ldots > j_{m-2} \ge 2}^{m} z^{2d_{j_1} + d_{j_2} + \ldots + d_{j_{m-2}}}, \quad I_{m,m-1}(z) = z^{\sigma_1} \sum_{j \ge 2}^{m} z^{d_j - m}\ , \quad (6.20)$$

$$J_{m,1}(z) = \sum_{j_1 > j_2 \ge 2}^{m} z^{d_{j_1} + d_{j_2}}, \ J_{m,2}(z) = 2 \sum_{j_1 > j_2 > j_3 \ge 2}^{m} z^{d_{j_1} + d_{j_2} + d_{j_3}}, \ldots, \ J_{m,m-2}(z) = (m-2) z^{\sigma_1 - m}.$$

Substituting the degrees of monomials (6.18) into two identities of Theorem 1 in accordance with an ordinal number k of syzygies, which monomials (6.18) belong to, we obtain Corollary.

Corollary 4 *Let a set $\{d_2, \ldots, d_m\}$ of $m-1$ distinct positive integers d_j be given. Then for $1 \le k \le m-2$ the following identities hold,*

$$2^k \sum_{j_1 \ge 2}^{m} d_{j_1}^k + \sum_{j_1 > j_2 \ge 2}^{m} (d_{j_1} + d_{j_2})^k - \sum_{j_1 \neq j_2 \ge 2}^{m} (2d_{j_1} + d_{j_2})^k - 2 \sum_{j_1 > j_2 > j_3 \ge 2}^{m} (d_{j_1} + d_{j_2} + d_{j_3})^k + \\ \ldots + (-1)^m \sum_{j \ge 2} (\sigma_1 + d_j - m)^k = 0\ ,$$

where $\sigma_1 = m + \sum_{j=2}^m d_j$. If $k = m-1$, then

$$2^{m-1} \sum_{j_1 \ge 2}^{m} d_{j_1}^{m-1} + \sum_{j_1 > j_2 \ge 2}^{m} (d_{j_1} + d_{j_2})^{m-1} - \sum_{j_1 \neq j_2 \ge 2}^{m} (2d_{j_1} + d_{j_2})^{m-1} - \\ 2 \sum_{j_1 > j_2 > j_3 \ge 2}^{m} (d_{j_1} + d_{j_2} + d_{j_3})^{m-1} + \ldots + (-1)^m \sum_{j \ge 2} (\sigma_1 + d_j - m)^{m-1} = (-1)^m m! \prod_{j=2}^{m} d_j\ .$$

As an example of the MED semigroup one can take the semigroup $\mathsf{S}(\{3,5,7\})$ with Hilbert series given in (2.8). A straightforward calculation shows that the polynomial identities of Corollary 4 are satisfied as well as the quasipolynomial identities of Theorem 2.

6.4.1 Almost Symmetric Semigroups of Maximal edim

Almost symmetric semigroups $\mathsf{S}(\mathbf{d}^m)$ were introduced in [4] as a generalization of the symmetric and pseudosymmetric ones. They can be defined [4] by equality $\beta_{m-1}(\mathbf{d}^m) = 1 + \#\Delta_{\mathcal{H}}(\mathbf{d}^m)$, where $\Delta_{\mathcal{H}}(\mathbf{d}^m) = \{h \notin \mathsf{S}(\mathbf{d}^m) \mid F(\mathbf{d}^m) - h \notin \mathsf{S}(\mathbf{d}^m)\}$ is a subset of the set of the gaps $\Delta(\mathbf{d}^m)$. A detailed study of such semigroups can be found in [4] and [17]. Here we discuss a special class of almost symmetric MED semigroups. Necessary and sufficient conditions for a minimal set $\mathbf{d}^m$ to generate such semigroup is that for every element $d_j \in \mathbf{d}^m$ there exists its counterpartner d_{m-j+1} such that (see [17], Theorem 7),

$$d_j + d_{m-j+1} = \frac{2\sigma_1}{m}\,, \qquad 1 \le j \le m\,. \tag{6.21}$$

Relations (6.21) do not reduce the total number of the polynomial identities in Corollary 4. However, they do simplify a large number of terms contributing to these identities.

Below we illustrate the polynomial identities of Corollary 4 by presenting the Hilbert series of almost symmetric MED semigroups $\mathsf{S}(\{4, 10, 19, 25\})$ taken from [4]. Its corresponding Betti numbers read: $\beta_1 = 6$, $\beta_2 = 8$, $\beta_3 = 3$, while the numerator of its Hilbert series was calculated in [17],

$$H(z) = \frac{1 - z^{20} - z^{29} - z^{35} - z^{38} + z^{39} - z^{44} + z^{45} + z^{48} - z^{50} + 2z^{54} + z^{60} + z^{63} - z^{64} + z^{69} - z^{73} - z^{79}}{(1 - z^4)(1 - z^{10})(1 - z^{19})(1 - z^{25})}\,.$$

Straightforward calculation shows that the polynomial identities in Corollary 4 are satisfied as well as the quasipolynomial identities of Theorem 2.

A Appendix: Combinatorial proof of identities (6.2) and (6.3)

Consider n sets of elements $a_{j,r} \in \mathbb{E}_r$, $\#\mathbb{E}_r = e_r$, $1 \le r \le n$, of different colors and construct a tuple $\mathbf{a}^k$ of length k which is composed of elements $a_{j,r}$. Denote by $\mathbb{C}_i$ a set of all $\mathbf{a}^k$-tuples which do not contain any element of the ith color, i.e.,

$$\mathbb{C}_i = \{\mathbf{a}^k \mid a_{j,r} \in \mathbf{a}^k,\ r \ne i\}\,, \quad \#\mathbb{C}_i = (E - e_i)^k\,, \quad E = \sum_{j=1}^{n} e_j\,. \tag{A1}$$

Consider an intersection of two such sets $\mathbb{C}_i$ and $\mathbb{C}_l$ which do not contain any element of the ith and lth colors,

$$\mathbb{C}_i \cap \mathbb{C}_l = \{\mathbf{a}^k \mid a_{j,r} \in \mathbf{a}^k,\ r \ne i,\ r \ne l\}\,, \quad \#\{\mathbb{C}_i \cap \mathbb{C}_l\} = (E - e_i - e_l)^k\,. \tag{A2}$$

Continuing to build intersections of three and more sets $\mathbb{C}_i$ write by the inclusion-exclusion principle an identity for their cardinalities,

$$\#\bigcup_{i=1}^{n} \mathbb{C}_i = \sum_{i=1}^{n} (E - e_i)^k - \sum_{i>l=1}^{n} (E - e_i - e_l)^k + \ldots - (-1)^n \sum_{i>l=1}^{n} (e_i + e_l)^k + (-1)^n \sum_{i=1}^{n} e_i^k, \tag{A3}$$

where $\bigcup_{i=1}^n \mathbb{C}_i = \{\mathbf{a}^k \mid a_{j,r} \in \mathbf{a}^k,\ 1 \le r \le n\}$. If $k < n$ then $\#\bigcup_{i=1}^n \mathbb{C}_i = E^k$ and therefore

$$\sum_{i=1}^n e_i^k - \sum_{i>l=1}^n (e_i+e_l)^k + \ldots + (-1)^n \sum_{i=1}^n (E-e_i)^k - (-1)^n E^k = 0\,. \tag{A4}$$

In the case $k = n$ we have $\#\bigcup_{i=1}^n \mathbb{C}_i = E^n - n!\prod_{i=1}^n e_i$. Inserting the last into (A3) we get

$$\sum_{i=1}^n e_i^n - \sum_{i>l=1}^n (e_i+e_l)^n + \ldots + (-1)^n \sum_{i=1}^n (E-e_i)^n - (-1)^n E^n = -(-1)^n n! \prod_{i=1}^n e_i\,. \tag{A5}$$

Substituting $n = m-1$ into (A4) and (A5) we arrive at (6.2) and (6.3), respectively.

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